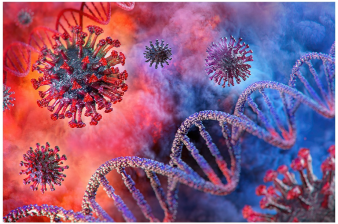
Data-Driven Stochastic Control via Non-i.i.d. Trajectories: Foundations and Guarantees

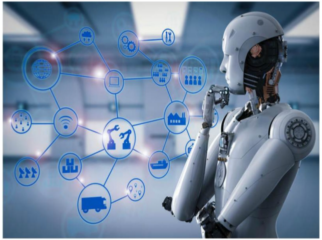
Abolfazl Lavaei

Newcastle University, UK

Email: abolfazl.lavaei@newcastle.ac.uk
Homepage: www.lavaei-cps.com

Stochastic Control Systems





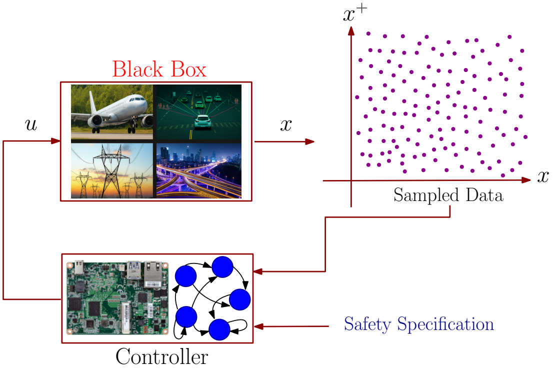
Stochastic Control: Challenges

- **Potential difficulties:**
  - Stochastic nature of dynamics

Stochastic Control: Challenges

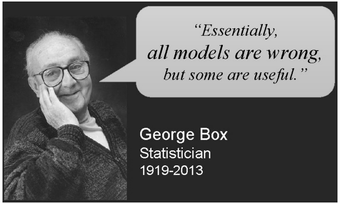
- **Potential difficulties:**
  - Stochastic nature of dynamics
  - Lack of mathematical models

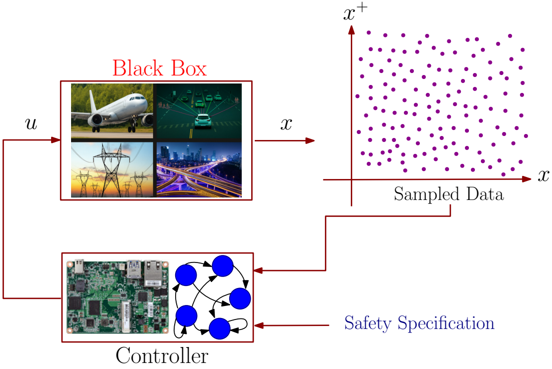
Data-Driven Analysis with Provable Guarantees



Data-Driven Analysis with Provable Guarantees

- Closed-form models: not available or too complex to deal with
- Model-based techniques cannot be useful

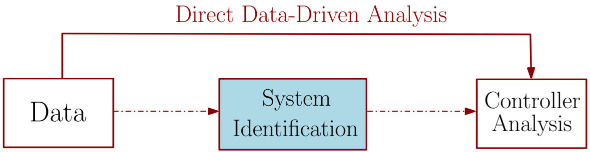




Data-Driven Control: Indirect vs. Direct Methods

- Indirect data-driven techniques: System identification followed by model-based methods
  - Two-level computational complexity

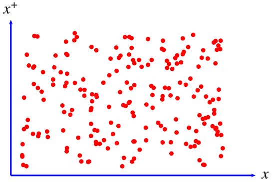
Data-Driven Control: Indirect vs. Direct Methods



- Indirect data-driven techniques: System identification followed by model-based methods
  - Two-level computational complexity
- Direct data-driven techniques: Directly employ system measurements
  - More samples for robustness guarantees

Direct Data-Driven Control: Scenario-based vs. Trajectory-based Approaches

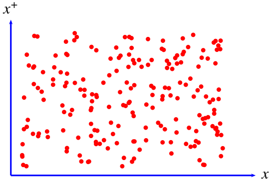
Scenario-based Approach



- i.i.d. samples

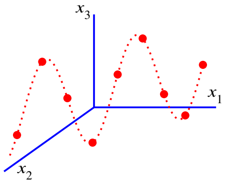
Direct Data-Driven Control: Scenario-based vs. Trajectory-based Approaches

Scenario-based Approach



- i.i.d. samples

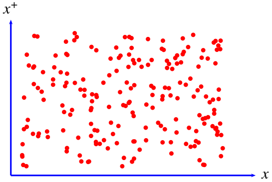
Trajectory-based Approach



- One set of (non-i.i.d.) time-series data

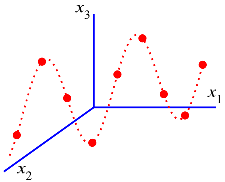
Direct Data-Driven Control: Scenario-based vs. Trajectory-based Approaches

Scenario-based Approach



- i.i.d. samples

Trajectory-based Approach

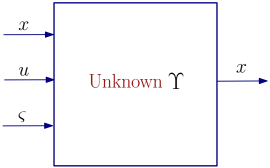


- One set of (non-i.i.d.) time-series data

Question of interest

How to design safety controllers with probabilistic confidence using trajectory-based approaches for stochastic control systems with unknown models?

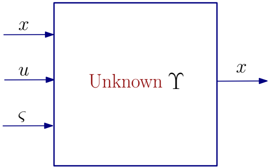
Discrete-time Nonlinear Polynomial Systems (dt-NPS)



$\Upsilon : x^+ = f(x) + g(x)u + \varsigma$

- $x \in X \subseteq \mathbb{R}^n$ and $u \in U \subseteq \mathbb{R}^m$
- $f : X \to X$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$
- $\varsigma \in Z \subseteq \mathbb{R}^n$, where $\|\varsigma\| \leq \varpi \in \mathbb{R}_0^+$

Discrete-time Nonlinear Polynomial Systems (dt-NPS)



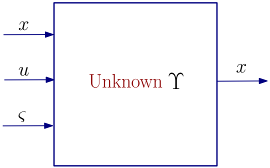
$\Upsilon: x^+ = f(x) + g(x)u + \varsigma$

- $x \in X \subseteq \mathbb{R}^n$ and $u \in U \subseteq \mathbb{R}^m$
- $f: X \to X$ and $g: \mathbb{R}^n \to \mathbb{R}^{n \times m}$
- $\varsigma \in Z \subseteq \mathbb{R}^n$, where $\|\varsigma\| \leq \varpi \in \mathbb{R}_0^+$

$\Upsilon: x^+ = A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma$

- $A \in \mathbb{R}^{n \times l}$ and $B \in \mathbb{R}^{n \times q}$
- $\mathcal{F}(x) \in \mathbb{R}^l$, with $\mathcal{F}(\mathbf{0_n}) = \mathbf{0_l}$
- $\mathcal{G}(x) \in \mathbb{R}^{q \times m}$

Discrete-time Nonlinear Polynomial Systems (dt-NPS)



$\Upsilon: x^+ = f(x) + g(x)u + \varsigma$

- $x \in X \subseteq \mathbb{R}^n$ and $u \in U \subseteq \mathbb{R}^m$
- $f: X \to X$ and $g: \mathbb{R}^n \to \mathbb{R}^{n\times m}$
- $\varsigma \in Z \subseteq \mathbb{R}^n$, where $\|\varsigma\| \leq \varpi \in \mathbb{R}_0^+$

$\Upsilon: x^+ = A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma$

- $A \in \mathbb{R}^{n\times l}$ and $B \in \mathbb{R}^{n\times q}$
- $\mathcal{F}(x) \in \mathbb{R}^l$, with $\mathcal{F}(\mathbf{0}_n) = \mathbf{0}_l$
- $\mathcal{G}(x) \in \mathbb{R}^{q\times m}$

▷ A and B are unknown

▷ $\mathcal{F}(x)$ and $\mathcal{G}(x)$: Access to extended dictionary (*i.e.,* library or family of functions)

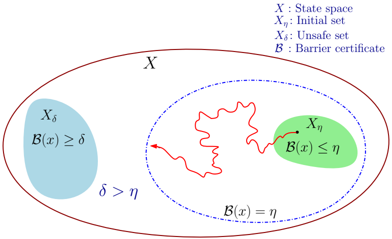
Robust Control Barrier Certificates (R-CBC) & Robust Safety Controller (R-SC)

R-CBC and R-SC[1]

Consider a dt-NPS Υ with $X_\eta, X_\delta \subseteq X$. Suppose there exist $\mathcal{B}: X \to \mathbb{R}_0^+$ and $\eta, \delta, \rho \in \mathbb{R}_0^+$, with $\delta > \eta$, and $\kappa \in (0,1)$, such that:

- $\forall x \in X_\eta$: $\quad \mathcal{B}(x) \leq \eta,$
- $\forall x \in X_\delta$: $\quad \mathcal{B}(x) \geq \delta,$
- $\forall x \in \tilde{X} = \{x \in X \colon \mathcal{B}(x) < \delta\}, \exists u \in U,$ such that $\forall \varsigma \in Z$:

$$\mathcal{B}(A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma) \leq \kappa \mathcal{B}(x) + \rho \|\varsigma\|^2.$$



[1] O. Akbarzadeh, M.H. Ashoori, and A. Lavaei, "Learning Robust Safety Controllers for Uncertain Input-Affine Polynomial Systems", *CDC 2025*.

Robust Control Barrier Certificates (R-CBC) & Robust Safety Controller (R-SC)

R-CBC and R-SC[1]

Consider a dt-NPS Υ with $X_\eta, X_\delta \subseteq X$. Suppose there exist $\mathcal{B}: X \to \mathbb{R}_0^+$ and $\eta, \delta, \rho \in \mathbb{R}_0^+$, with $\delta > \eta$, and $\kappa \in (0,1)$, such that:

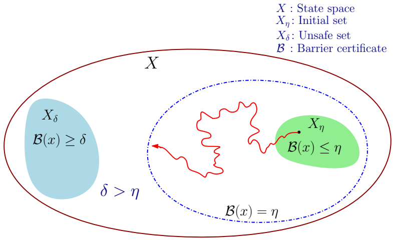
- $\forall x \in X_\eta$: $\quad \mathcal{B}(x) \leq \eta,$
- $\forall x \in X_\delta$: $\quad \mathcal{B}(x) \geq \delta,$
- $\forall x \in \tilde{X} = \{x \in X \colon \mathcal{B}(x) < \delta\}, \exists u \in U,$ such that $\forall \varsigma \in Z$:

$$\mathcal{B}(A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma) \leq \kappa\mathcal{B}(x) + \rho\|\varsigma\|^2.$$

If

$$\rho\varpi \leq \delta(1-\kappa),$$

then $x_{x_0uw}(k) \notin X_\delta$ for any $x_0 \in X_0$ and $k \in \mathbb{N}$ under signals $u(\cdot)$ and $\varsigma(\cdot)$.



[1] O. Akbarzadeh, M.H. Ashoori, and A. Lavaei, "Learning Robust Safety Controllers for Uncertain Input-Affine Polynomial Systems", *CDC 2025*.

Data-Driven Design of R-CBC and R-SC

- $\overrightarrow{\mathbb{X}} = [x(1) \quad x(2) \quad \ldots \quad x(T)]$
- $\mathbb{X} = [x(0) \quad x(1) \quad \ldots \quad x(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \ldots \quad u(T-1)]$
- $\mathbb{Z} = [\varsigma(0) \quad \varsigma(1) \quad \ldots \quad \varsigma(T-1)]$

Data-Driven Design of R-CBC and R-SC

- $\overrightarrow{\mathbb{X}} = [x(1) \quad x(2) \quad \dots \quad x(T)]$
- $\mathbb{X} = [x(0) \quad x(1) \quad \dots \quad x(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \dots \quad u(T-1)]$
- $\mathbb{Z} = [\varsigma(0) \quad \varsigma(1) \quad \dots \quad \varsigma(T-1)]$
- $\mathbb{F} = [\mathcal{F}(x(0)) \;\; \mathcal{F}(x(1)) \; \dots \; \mathcal{F}(x(T-1))]$
- $\mathbb{G} = [\mathcal{G}(x(0))u(0) \;\; \mathcal{G}(x(1))u(1) \; \dots \; \mathcal{G}(x(T-1))u(T-1))]$

Data-Driven Design of R-CBC and R-SC

- $\overrightarrow{\mathbb{X}} = [x(1) \quad x(2) \quad \dots \quad x(T)]$
- $\mathbb{X} = [x(0) \quad x(1) \quad \dots \quad x(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \dots \quad u(T-1)]$
- $\mathbb{Z} = [\varsigma(0) \quad \varsigma(1) \quad \dots \quad \varsigma(T-1)]$
- $\mathbb{F} = [\mathcal{F}(x(0)) \ \ \mathcal{F}(x(1)) \ \dots \ \mathcal{F}(x(T-1))]$
- $\mathbb{G} = [\mathcal{G}(x(0))u(0) \ \ \mathcal{G}(x(1))u(1) \ \dots \ \mathcal{G}(x(T-1))u(T-1))]$

▷ **R-CBC:** $\mathcal{B}(x) = x^\top P x$

▷ **Robust Safety Controller:** $u = \mathcal{K}(x)x$

Data-Driven Design of R-CBC and R-SC

- $\overrightarrow{\mathbb{X}} = [x(1) \quad x(2) \quad \dots \quad x(T)]$
- $\mathbb{X} = [x(0) \quad x(1) \quad \dots \quad x(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \dots \quad u(T-1)]$
- $\mathbb{Z} = [\varsigma(0) \quad \varsigma(1) \quad \dots \quad \varsigma(T-1)]$
- $\mathbb{F} = [\mathcal{F}(x(0)) \ \ \mathcal{F}(x(1)) \ \dots \ \mathcal{F}(x(T-1))]$
- $\mathbb{G} = [\mathcal{G}(x(0))u(0) \ \ \mathcal{G}(x(1))u(1) \ \dots \ \mathcal{G}(x(T-1))u(T-1))]$

▷ **R-CBC:** $\mathcal{B}(x) = x^\top P x$

▷ **Robust Safety Controller:** $u = \mathcal{K}(x)x$

$$\overrightarrow{\mathbb{X}}_j = A\mathbb{F}_j + B\mathbb{G}_j + \mathbb{Z}_j = \Phi\mathbb{H}_j + \mathbb{Z}_j, \ j \in \{1, \dots, T\}$$

where $\Phi = [A \quad B]$, $\mathbb{H}_j = \begin{bmatrix} \mathbb{F}_j \\ \mathbb{G}_j \end{bmatrix}$

▷ $\mathbb{Z}_j = \overrightarrow{\mathbb{X}}_j - \Phi\mathbb{H}_j$

Data-Driven Design of R-CBC and R-SC

- $\overrightarrow{\mathbb{X}} = [x(1) \quad x(2) \quad \dots \quad x(T)]$
- $\mathbb{X} = [x(0) \quad x(1) \quad \dots \quad x(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \dots \quad u(T-1)]$
- $\mathbb{Z} = [\varsigma(0) \quad \varsigma(1) \quad \dots \quad \varsigma(T-1)]$
- $\mathbb{F} = [\mathcal{F}(x(0)) \ \ \mathcal{F}(x(1)) \ \dots \ \mathcal{F}(x(T-1))]$
- $\mathbb{G} = [\mathcal{G}(x(0))u(0) \ \ \mathcal{G}(x(1))u(1) \ \dots \ \mathcal{G}(x(T-1))u(T-1))]$

▷ **R-CBC:** $\mathcal{B}(x) = x^\top P x$

▷ **Robust Safety Controller:** $u = \mathcal{K}(x)x$

$\overrightarrow{\mathbb{X}}_j = A\mathbb{F}_j + B\mathbb{G}_j + \mathbb{Z}_j = \Phi\mathbb{H}_j + \mathbb{Z}_j, \ j \in \{1, \dots, T\}$ where $\Phi = [A \quad B], \quad \mathbb{H}_j = \begin{bmatrix} \mathbb{F}_j \\ \mathbb{G}_j \end{bmatrix}$

▷ $\mathbb{Z}_j = \overrightarrow{\mathbb{X}}_j - \Phi\mathbb{H}_j$

$\mathbb{Z}_j\mathbb{Z}_j^\top \preceq \varpi^2\mathbb{I}_n$

▷ $(\overrightarrow{\mathbb{X}}_j - \Phi\mathbb{H}_j)(\overrightarrow{\mathbb{X}}_j - \Phi\mathbb{H}_j)^\top - \varpi^2\mathbb{I}_n \preceq 0 \quad \rightarrow \quad \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix}^\top \begin{bmatrix} \overrightarrow{\mathbb{X}}_j\overrightarrow{\mathbb{X}}_j^\top - \varpi^2\mathbb{I}_n & -\overrightarrow{\mathbb{X}}_j\mathbb{H}_j^\top \\ * & \mathbb{H}_j\mathbb{H}_j^\top \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix} \preceq 0$

Data-Driven Design of R-CBC and R-SC: Main Result

Main Result: R-CBC and R-SC Design

Given an unknown dt-NPS Υ, let there exist $\bar{\eta}, \bar{\delta}, \in \mathbb{R}^+$, with $\bar{\eta} > \bar{\delta}$, $\kappa \in (0,1)$, matrices $\bar{P} \succ 0$ and $\bar{\mathcal{K}}(x)$, and $\alpha_{j=1,\dots,T}(x)$: $\mathbb{R}^n \to \mathbb{R}_0^+$, such that

- $\bar{P} - \bar{\eta} z_\eta z_\eta^\top \succeq 0, \qquad \forall x \in X_\eta,$ with $X_\eta \subseteq \{x \in \mathbb{R}^n \colon \ xx^\top \preceq z_\eta z_\eta^\top, \ z_\eta \in \mathbb{R}^n\}$,
- $\bar{P} - \bar{\delta} z_\delta z_\delta^\top \preceq 0, \qquad \forall x \in X_\delta,$ with $X_\delta \subseteq \{x \in \mathbb{R}^n \colon \ xx^\top \succeq z_\delta z_\delta^\top, \ z_\delta \in \mathbb{R}^n\}$,
- $$\begin{bmatrix} -\kappa \bar{P} & 0 & 0 \\ * & 0 & \begin{bmatrix} \mathcal{J}(x)\bar{P} \\ \mathcal{G}(x)\bar{\mathcal{K}}(x) \end{bmatrix} \\ * & * & -(1+\mu)^{-1}\bar{P} \end{bmatrix} - \sum_{j=1}^{T} \alpha_j(x) \begin{bmatrix} \mathcal{R}^{DC_j} & 0 \\ * & 0 \end{bmatrix} \preceq 0, \qquad \forall x \in X,$$

with

$$\mathcal{R}^{DC_j} = \begin{bmatrix} \overrightarrow{\mathbb{X}}_j \overrightarrow{\mathbb{X}}_j^\top - \varpi^2 \mathbb{I}_n & -\overrightarrow{\mathbb{X}}_j \mathbb{H}_j^\top \\ * & \mathbb{H}_j \mathbb{H}_j^\top \end{bmatrix}.$$

Data-Driven Design of R-CBC and R-SC: Main Result

Main Result: R-CBC and R-SC Design

Given an unknown dt-NPS Υ, let there exist $\bar{\eta}, \bar{\delta}, \in \mathbb{R}^+$, with $\bar{\eta} > \bar{\delta}$, $\kappa \in (0,1)$, matrices $\bar{P} \succ 0$ and $\bar{\mathcal{K}}(x)$, and $\alpha_{j=1,\dots,T}(x)$: $\mathbb{R}^n \to \mathbb{R}_0^+$, such that

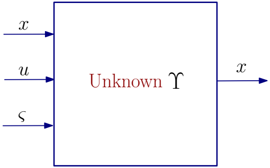
- $\bar{P} - \bar{\eta} z_\eta z_\eta^\top \succeq 0, \qquad \forall x \in X_\eta,$ with $X_\eta \subseteq \{x \in \mathbb{R}^n\colon\ xx^\top \preceq z_\eta z_\eta^\top,\ z_\eta \in \mathbb{R}^n\}$,
- $\bar{P} - \bar{\delta} z_\delta z_\delta^\top \preceq 0, \qquad \forall x \in X_\delta,$ with $X_\delta \subseteq \{x \in \mathbb{R}^n\colon\ xx^\top \succeq z_\delta z_\delta^\top,\ z_\delta \in \mathbb{R}^n\}$,
- $\begin{bmatrix} -\kappa \bar{P} & 0 & 0 \\ * & 0 & \begin{bmatrix} \mathcal{J}(x)\bar{P} \\ \mathcal{G}(x)\bar{\mathcal{K}}(x) \end{bmatrix} \\ * & * & -(1+\mu)^{-1}\bar{P} \end{bmatrix} - \sum_{j=1}^{T} \alpha_j(x) \begin{bmatrix} \mathcal{R}^{DC_j} & 0 \\ * & 0 \end{bmatrix} \preceq 0, \qquad \forall x \in X,$

with

$$\mathcal{R}^{DC_j} = \begin{bmatrix} \overrightarrow{\mathbb{X}}_j \overrightarrow{\mathbb{X}}_j^\top - \varpi^2 \mathbb{I}_n & -\overrightarrow{\mathbb{X}}_j \mathbb{H}_j^\top \\ * & \mathbb{H}_j \mathbb{H}_j^\top \end{bmatrix}.$$

Then, $\mathcal{B}(x) = x^\top P x$, with $P = \bar{P}^{-1}$, and $u = \mathcal{K}(x)x$, with $\mathcal{K}(x) = \bar{\mathcal{K}}(x)\bar{P}^{-1} = \bar{\mathcal{K}}(x)P$. In addition, $\eta = \bar{\eta}^{-1}$, $\delta = \bar{\delta}^{-1}$ (where $\eta < \delta$), and $\rho = (1+\mu^{-1})\|\sqrt{P}\|^2$.

Discrete-time Stochastic Nonlinear Polynomial Systems (dt-SNPS)

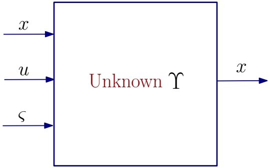


$\Upsilon: x^{+} = A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma$

- $A \in \mathbb{R}^{n\times l}$ and $B \in \mathbb{R}^{n\times q}$
- $\mathcal{F}(x) \in \mathbb{R}^{l}$, with $\mathcal{F}(\mathbf{0}_{n}) = \mathbf{0}_{l}$
- $\mathcal{G}(x) \in \mathbb{R}^{q\times m}$

- $(\Omega, \mathbb{F}_{\Omega}, \mathbb{P}_{\Omega})$
- $\varsigma := \{\varsigma(k) : \Omega \to \mathcal{V}_{\varsigma},\ k \in \mathbb{N}\}$;
- $\mu\mu^{\top} \preceq \Gamma_{\mu}$ and $\Sigma \preceq \Gamma_{\Sigma}$

Discrete-time Stochastic Nonlinear Polynomial Systems (dt-SNPS)



$\Upsilon: x^+ = A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma$

- $A \in \mathbb{R}^{n\times l}$ and $B \in \mathbb{R}^{n\times q}$
- $\mathcal{F}(x) \in \mathbb{R}^{l}$, with $\mathcal{F}(\mathbf{0}_{\boldsymbol{n}}) = \mathbf{0}_{\boldsymbol{l}}$
- $\mathcal{G}(x) \in \mathbb{R}^{q\times m}$

- $(\Omega, \mathbb{F}_\Omega, \mathbb{P}_\Omega)$
- $\varsigma := \{\varsigma(k) : \Omega \to \mathcal{V}_\varsigma,\ k \in \mathbb{N}\}$;
- $\mu\mu^\top \preceq \Gamma_\mu$ and $\Sigma \preceq \Gamma_\Sigma$

▷ A and B are unknown

▷ $\mathcal{F}(x)$ and $\mathcal{G}(x)$: Access to extended dictionary (*i.e.,* library or family of functions)

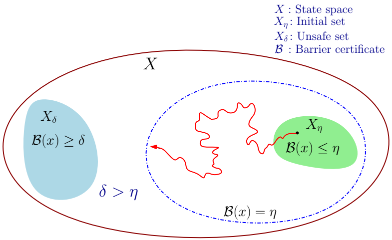
Stochastic Control Barrier Certificates (S-CBC) & Safety Controller

S-CBC and Safety Controller[2]

Consider a dt-NPS Υ with $X_\eta, X_\delta \subseteq X$. Suppose there exist $\mathcal{B} : X \to \mathbb{R}_0^+$ and $\eta, \delta, \psi \in \mathbb{R}_0^+$, with $\delta > \eta$, and $\kappa \in (0,1)$, such that:

- $\forall x \in X_\eta$: $\quad \mathcal{B}(x) \leq \eta,$
- $\forall x \in X_\delta$: $\quad \mathcal{B}(x) \geq \delta,$
- $\forall x \in X, \exists u \in U$, such that:

$$\mathbb{E}\Big[\mathcal{B}(A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma) \mid x, u\Big] \leq \kappa \mathcal{B}(x) + \psi.$$



[2] A. Lavaei, “Data-Driven Stochastic Control via Non-i.i.d. Trajectories: Foundations and Guarantees", *IEEE TAC*, under review, 2025.

Stochastic Control Barrier Certificates (S-CBC) & Safety Controller

S-CBC and Safety Controller[2]

Consider a dt-NPS Υ with $X_\eta, X_\delta \subseteq X$. Suppose there exist $\mathcal{B}: X \to \mathbb{R}_0^+$ and $\eta, \delta, \psi \in \mathbb{R}_0^+$, with $\delta > \eta$, and $\kappa \in (0,1)$, such that:

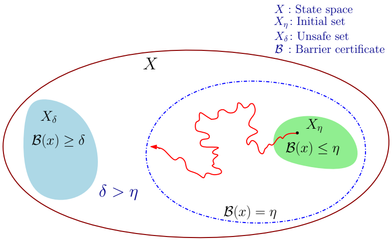
- $\forall x \in X_\eta$: $\quad \mathcal{B}(x) \leq \eta$,
- $\forall x \in X_\delta$: $\quad \mathcal{B}(x) \geq \delta$,
- $\forall x \in X, \exists u \in U$, such that:

$$\mathbb{E}\Big[\mathcal{B}(A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma) \mid x, u\Big] \leq \kappa\mathcal{B}(x) + \psi.$$

Then:

$$\mathbb{P}\Big\{x_{x_0 u} \notin X_\delta \text{ for all } k \in [0, \mathcal{T}] \mid x_0\Big\} \geq 1 - \beta_1,$$

where $\beta_1 = \frac{\eta + \psi\mathcal{T}}{\delta}$.



[2] A. Lavaei, "Data-Driven Stochastic Control via Non-i.i.d. Trajectories: Foundations and Guarantees", *IEEE TAC*, under review, 2025.

Data-Driven Design of S-CBC and Safety Controller

For all $i \in \{1, 2, \dots, N\}$:

- $\overrightarrow{\mathbb{X}}^i = [x^i(1) \quad x^i(2) \quad \dots \quad x^i(T)]$
- $\mathbb{X}^i = [x(0) \quad x^i(1) \quad \dots \quad x^i(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \dots \quad u(T-1)]$
- $\mathbb{Z}^i = [\varsigma^i(0) \quad \varsigma^i(1) \quad \dots \quad \varsigma^i(T-1)]$

Data-Driven Design of S-CBC and Safety Controller

For all $i \in \{1, 2, \ldots, N\}$:

- $\overrightarrow{\mathbb{X}}^i = [x^i(1) \quad x^i(2) \quad \ldots \quad x^i(T)]$
- $\mathbb{X}^i = [x(0) \quad x^i(1) \quad \ldots \quad x^i(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \ldots \quad u(T-1)]$
- $\mathbb{Z}^i = [\varsigma^i(0) \quad \varsigma^i(1) \quad \ldots \quad \varsigma^i(T-1)]$
- $\mathbb{F}^i = [\mathcal{F}(x(0)) \;\; \mathcal{F}(x^i(1)) \; \ldots \; \mathcal{F}(x^i(T-1))]$
- $\mathbb{G}^i = [\mathcal{G}(x(0))u(0) \;\; \mathcal{G}(x^i(1))u(1) \; \ldots \; \mathcal{G}(x^i(T-1))u(T-1))]$

Data-Driven Design of S-CBC and Safety Controller

For all $i \in \{1, 2, \dots, N\}$:

- $\overrightarrow{\mathbb{X}}^i = [x^i(1) \quad x^i(2) \quad \dots \quad x^i(T)]$
- $\mathbb{X}^i = [x(0) \quad x^i(1) \quad \dots \quad x^i(T-1)]$
- $\mathbb{U} = [u(0) \quad u(1) \quad \dots \quad u(T-1)]$
- $\mathbb{Z}^i = [\varsigma^i(0) \quad \varsigma^i(1) \quad \dots \quad \varsigma^i(T-1)]$
- $\mathbb{F}^i = [\mathcal{F}(x(0)) \;\; \mathcal{F}(x^i(1)) \; \dots \; \mathcal{F}(x^i(T-1))]$
- $\mathbb{G}^i = [\mathcal{G}(x(0))u(0) \;\; \mathcal{G}(x^i(1))u(1) \; \dots \; \mathcal{G}(x^i(T-1))u(T-1))]$

Problem of interest

Given an unknown dt-SNPS with process noise ς, design an S-CBC and its safety controller, while quantifying the probabilistic safety level $\beta_1 \in (0, 1]$ and confidence level $\beta_2 \in (0, 1]$ purely based on data, *i.e.,*

$$\mathbb{P}\Big\{\mathbb{P}\{\Upsilon \vDash \mathbb{S}\} \geq 1 - \beta_1\Big\} \geq 1 - \beta_2.$$

Data-Driven Design of S-CBC and Safety Controller

$$\mathbb{E}\left[\varsigma\varsigma^\top\right] = \Sigma + \mu\mu^\top \preceq \Gamma_\Sigma + \Gamma_\mu$$

Data-Driven Design of S-CBC and Safety Controller

$$\mathbb{E}\left[\varsigma\varsigma^\top\right] = \Sigma + \mu\mu^\top \preceq \Gamma_\Sigma + \Gamma_\mu$$

Lemma 1: Formalizing Empirical Approximation

Let $\mathbb{Z}^1_{(\cdot)}, \ldots, \mathbb{Z}^N_{(\cdot)} \in \mathbb{R}^n$ be N independent samples drawn from $\varsigma \in \mathbb{R}^n$, where $\mu\mu^\top \preceq \Gamma_\mu$ and $\Sigma \preceq \Gamma_\Sigma$. Then

$$\mathbb{P}\left(\left\| \frac{1}{N} \sum_{i=1}^{N} \mathbb{Z}^i_{(\cdot)} \mathbb{Z}^{i\top}_{(\cdot)} - \mathbb{E}[\varsigma\varsigma^\top] \right\| < \epsilon\right) \geq 1 - \bar{\beta}_2,$$

with $\bar{\beta}_2 = \frac{1}{N\epsilon^2}\big(\operatorname{Tr}(\Gamma_\Sigma^2) + (\operatorname{Tr}(\Gamma_\Sigma))^2 + 2\lambda_{\max}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu) + 2\operatorname{Tr}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu)\big)$.

Data-Driven Design of S-CBC and Safety Controller

$\mathbb{E}\left[\varsigma\varsigma^\top\right] = \Sigma + \mu\mu^\top \preceq \Gamma_\Sigma + \Gamma_\mu$

Lemma 1: Formalizing Empirical Approximation

Let $\mathbb{Z}^1_{(\cdot)}, \dots, \mathbb{Z}^N_{(\cdot)} \in \mathbb{R}^n$ be N independent samples drawn from $\varsigma \in \mathbb{R}^n$, where $\mu\mu^\top \preceq \Gamma_\mu$ and $\Sigma \preceq \Gamma_\Sigma$. Then

$$\mathbb{P}\left(\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbb{Z}^i_{(\cdot)}\mathbb{Z}^{i\top}_{(\cdot)} - \mathbb{E}[\varsigma\varsigma^\top]\right\| < \epsilon\right) \geq 1 - \bar{\beta}_2,$$

with $\bar{\beta}_2 = \frac{1}{N\epsilon^2}\left(\operatorname{Tr}(\Gamma_\Sigma^2) + (\operatorname{Tr}(\Gamma_\Sigma))^2 + 2\lambda_{\max}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu) + 2\operatorname{Tr}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu)\right)$.

Lemma 2: Data-Conformity Constraint

Under Lemma 1, the following statement holds true with a confidence of at least $1 - \bar{\beta}_2$:

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{Z}^i_{(\cdot)}\mathbb{Z}^{i\top}_{(\cdot)} \preceq \Gamma_\Sigma + \Gamma_\mu + \epsilon\mathbb{I}_n$$

Data-Driven Design of S-CBC and Safety Controller

$$\mathbb{E}\left[\varsigma\varsigma^\top\right] = \Sigma + \mu\mu^\top \preceq \Gamma_\Sigma + \Gamma_\mu$$

Lemma 1: Formalizing Empirical Approximation

Let $\mathbb{Z}^1_{(\cdot)}, \ldots, \mathbb{Z}^N_{(\cdot)} \in \mathbb{R}^n$ be N independent samples drawn from $\varsigma \in \mathbb{R}^n$, where $\mu\mu^\top \preceq \Gamma_\mu$ and $\Sigma \preceq \Gamma_\Sigma$. Then

$$\mathbb{P}\left(\left\| \frac{1}{N} \sum_{i=1}^{N} \mathbb{Z}^i_{(\cdot)} \mathbb{Z}^{i\top}_{(\cdot)} - \mathbb{E}[\varsigma\varsigma^\top] \right\| < \epsilon \right) \geq 1 - \bar{\beta}_2,$$

with $\bar{\beta}_2 = \frac{1}{N\epsilon^2}\left(\operatorname{Tr}(\Gamma_\Sigma^2) + (\operatorname{Tr}(\Gamma_\Sigma))^2 + 2\lambda_{\max}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu) + 2\operatorname{Tr}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu)\right)$.

Lemma 2: Data-Conformity Constraint

Under Lemma 1, the following statement holds true with a confidence of at least $1 - \bar{\beta}_2$:

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{Z}^i_{(\cdot)} \mathbb{Z}^{i\top}_{(\cdot)} \preceq \Gamma_\Sigma + \Gamma_\mu + \epsilon \mathbb{I}_n$$

$$\rightarrow \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix}^\top \begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N} \overrightarrow{\mathbb{X}}^i_j \overrightarrow{\mathbb{X}}^{i\top}_j - (\Gamma_\Sigma + \Gamma_\mu + \epsilon\mathbb{I}_n) & -\frac{1}{N}\sum_{i=1}^{N} \overrightarrow{\mathbb{X}}^i_j \mathbb{H}^{i\top}_j \\ * & \frac{1}{N}\sum_{i=1}^{N} \mathbb{H}^i_j \mathbb{H}^{i\top}_j \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix} \preceq 0$$

Data-Driven Design of S-CBC and Safety Controller: Main Result

Main Result: S-CBC and Safety Controller Design

Given an unknown dt-NPS Υ, let there exist $\bar{\eta}, \bar{\delta}, \in \mathbb{R}^+$, with $\bar{\eta} > \bar{\delta}$, $\kappa \in (0,1)$, matrices $\bar{P} \succ 0$ and $\bar{\mathcal{K}}(x)$, and $\alpha_{j=1,\dots,T}(x)$: $\mathbb{R}^n \to \mathbb{R}_0^+$, such that

- $\bar{P} - \bar{\eta} z_\eta z_\eta^\top \succeq 0, \quad \forall x \in X_\eta,$ with $X_\eta \subseteq \{x \in \mathbb{R}^n\colon\ xx^\top \preceq z_\eta z_\eta^\top,\ z_\eta \in \mathbb{R}^n\}$,
- $\bar{P} - \bar{\delta} z_\delta z_\delta^\top \preceq 0, \quad \forall x \in X_\delta,$ with $X_\delta \subseteq \{x \in \mathbb{R}^n\colon\ xx^\top \succeq z_\delta z_\delta^\top,\ z_\delta \in \mathbb{R}^n\}$,
- $$\begin{bmatrix} -\kappa \bar{P} & 0 & 0 \\ * & 0 & \begin{bmatrix} \mathcal{J}(x)\bar{P} \\ \mathcal{G}(x)\bar{\mathcal{K}}(x) \end{bmatrix} \\ * & * & -(1+\rho)^{-1}\bar{P} \end{bmatrix} - \sum_{j=1}^{T} \alpha_j(x) \begin{bmatrix} \mathcal{R}^{DC_j} & 0 \\ * & 0 \end{bmatrix} \preceq 0, \qquad \forall x \in X,$$

with

$$\mathcal{R}^{DC_j} = \begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N} \overrightarrow{\mathbb{X}}_j^i \overrightarrow{\mathbb{X}}_j^{i\top} - (\Gamma_\Sigma + \Gamma_\mu + \epsilon \mathbb{I}_n) & -\frac{1}{N}\sum_{i=1}^{N} \overrightarrow{\mathbb{X}}_j^i \mathbb{H}_j^{i\top} \\ * & \frac{1}{N}\sum_{i=1}^{N} \mathbb{H}_j^i \mathbb{H}_j^{i\top} \end{bmatrix}, \text{ where } \mathbb{H}_j^i = \begin{bmatrix} \mathbb{F}_j^i \\ \mathbb{G}_j^i \end{bmatrix}.$$

Data-Driven Design of S-CBC and Safety Controller: Main Result

Main Result: S-CBC and Safety Controller Design

Given an unknown dt-NPS Υ, let there exist $\bar{\eta}, \bar{\delta}, \in \mathbb{R}^+$, with $\bar{\eta} > \bar{\delta}$, $\kappa \in (0,1)$, matrices $\bar{P} \succ 0$ and $\bar{\mathcal{K}}(x)$, and $\alpha_{j=1,\dots,T}(x)$: $\mathbb{R}^n \to \mathbb{R}_0^+$, such that

- $\bar{P} - \bar{\eta} z_\eta z_\eta^\top \succeq 0, \qquad \forall x \in X_\eta,$ with $X_\eta \subseteq \{x \in \mathbb{R}^n \colon\ xx^\top \preceq z_\eta z_\eta^\top,\ z_\eta \in \mathbb{R}^n\}$,
- $\bar{P} - \bar{\delta} z_\delta z_\delta^\top \preceq 0, \qquad \forall x \in X_\delta,$ with $X_\delta \subseteq \{x \in \mathbb{R}^n \colon\ xx^\top \succeq z_\delta z_\delta^\top,\ z_\delta \in \mathbb{R}^n\}$,
- $$\begin{bmatrix} -\kappa\bar{P} & 0 & 0 \\ * & 0 & \begin{bmatrix} \mathcal{J}(x)\bar{P} \\ \mathcal{G}(x)\bar{\mathcal{K}}(x) \end{bmatrix} \\ * & * & -(1+\rho)^{-1}\bar{P} \end{bmatrix} - \sum_{j=1}^{T} \alpha_j(x) \begin{bmatrix} \mathcal{R}^{DC_j} & 0 \\ * & 0 \end{bmatrix} \preceq 0, \qquad \forall x \in X,$$

with

$$\mathcal{R}^{DC_j} = \begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N} \overrightarrow{\mathbb{X}}_j^i \overrightarrow{\mathbb{X}}_j^{i\top} - (\Gamma_\Sigma + \Gamma_\mu + \epsilon\mathbb{I}_n) & -\frac{1}{N}\sum_{i=1}^{N} \overrightarrow{\mathbb{X}}_j^i \mathbb{H}_j^{i\top} \\ * & \frac{1}{N}\sum_{i=1}^{N} \mathbb{H}_j^i \mathbb{H}_j^{i\top} \end{bmatrix}, \text{ where } \mathbb{H}_j^i = \begin{bmatrix} \mathbb{F}_j^i \\ \mathbb{G}_j^i \end{bmatrix}.$$

Then, $\mathcal{B}(x) = x^\top P x$, with $P = \bar{P}^{-1}$, and $u = \mathcal{K}(x)x$, with $\mathcal{K}(x) = \bar{\mathcal{K}}(x)\bar{P}^{-1}$. In addition, $\eta = \bar{\eta}^{-1}$, $\delta = \bar{\delta}^{-1}$ $(\eta < \delta)$, and $\psi = (1+\rho^{-1})\,\mathrm{Tr}(P\Gamma_\mu) + \mathrm{Tr}(P\Gamma_\Sigma)$, with a confidence of at least $1 - \beta_2$, where $\beta_2 = T\bar{\beta}_2$.

Data-Driven Algorithm

Require: The state set X, extended dictionaries $\mathcal{F}(x), \mathcal{G}(x)$, and bounds $\Gamma_\mu, \Gamma_\Sigma$

1: Collect $\overrightarrow{\mathbb{X}}^i, \mathbb{X}^i$, and $\mathbb{U}$, where $i \in \{1, 2, \dots, N\}$
2: Construct $\mathbb{F}^i$ and $\mathbb{G}^i$
3: Initialize the desired ϵ and N, and compute $\bar{\beta}_2$
4: Initialize $\kappa \in (0, 1)$ and $\rho \in \mathbb{R}^+$
5: Solve conditions using SeDuMi and SOSTOOLS and compute $P = \bar{P}^{-1}$, $\mathcal{K}(x) = \bar{\mathcal{K}}(x)\bar{P}^{-1} = \bar{\mathcal{K}}(x)P$, $\eta = \bar{\eta}^{-1}$, and $\delta = \bar{\delta}^{-1}$
6: Compute ψ
7: Quantify β_1 and $\beta_2 = T\bar{\beta}_2$

Ensure: S-CBC $\mathcal{B}(x) = x^\top Px$, safety controller $u = \mathcal{K}(x)x$, and guaranteed probabilistic safety

$$\mathbb{P}\Big\{\mathbb{P}\{\Upsilon \vDash \mathbb{S}\} \geq 1 - \beta_1\Big\} \geq 1 - \beta_2$$

Discussion

- **Robust vs. Stochastic Analysis.** If $\varsigma \in Z = [-0.2, 0.2]^n$, then $\varpi^2 = 0.04n$ (yielding $0.04n\mathbb{I}_n$ in robust condition). In contrast, if $\varsigma \sim \mathcal{U}(\underbrace{-0.2}_{a}, \underbrace{0.2}_{b})^n$, then

$$\mathbb{E}[\varsigma\varsigma^\top] = \Sigma + \mu\mu^\top = \frac{(b-a)^2}{12}\mathbb{I}_n + (\frac{a+b}{2})^2 \mathbb{1}_n \mathbb{1}_n^\top + = \frac{0.16}{12}\mathbb{I}_n \approx 0.0133\mathbb{I}_n,$$

which is, even for $n = 1$, one-third of the worst-case bound 0.04 (*i.e.*, 66% less conservative).

Discussion

- **Robust vs. Stochastic Analysis.** If $\varsigma \in Z = [-0.2, 0.2]^n$, then $\varpi^2 = 0.04n$ (yielding $0.04n\mathbb{I}_n$ in robust condition). In contrast, if $\varsigma \sim \mathcal{U}(\underbrace{-0.2}_{a}, \underbrace{0.2}_{b})^n$, then

$$\mathbb{E}[\varsigma\varsigma^\top] = \Sigma + \mu\mu^\top = \frac{(b-a)^2}{12}\mathbb{I}_n + (\frac{a+b}{2})^2 \mathbb{1}_n \mathbb{1}_n^\top + = \frac{0.16}{12}\mathbb{I}_n \approx 0.0133\mathbb{I}_n,$$

which is, even for $n = 1$, one-third of the worst-case bound 0.04 (*i.e.,* 66% less conservative).

- **Trade-off Between ϵ and $\bar{\beta}_2$.** Demanding higher accuracy (*i.e.,* smaller ϵ) reduces the confidence level $(1 - \bar{\beta}_2)$. This closed-form relation gives a lower bound for N based on ϵ and $\bar{\beta}_2$ as

$$N \geq \frac{1}{\bar{\beta}_2 \epsilon^2}\left(\operatorname{Tr}(\Gamma_\Sigma^2) + (\operatorname{Tr}(\Gamma_\Sigma))^2 + 2\lambda_{\max}(\Gamma_\Sigma) \operatorname{Tr}(\Gamma_\mu) + 2 \operatorname{Tr}(\Gamma_\Sigma) \operatorname{Tr}(\Gamma_\mu)\right).$$

Discussion

- **Robust vs. Stochastic Analysis.** If $\varsigma \in Z = [-0.2, 0.2]^n$, then $\varpi^2 = 0.04n$ (yielding $0.04n\mathbb{I}_n$ in robust condition). In contrast, if $\varsigma \sim \mathcal{U}(\underbrace{-0.2}_{a}, \underbrace{0.2}_{b})^n$, then

$$\mathbb{E}[\varsigma\varsigma^\top] = \Sigma + \mu\mu^\top = \frac{(b-a)^2}{12}\mathbb{I}_n + (\frac{a+b}{2})^2 \mathbb{1}_n \mathbb{1}_n^\top + = \frac{0.16}{12}\mathbb{I}_n \approx 0.0133\mathbb{I}_n,$$

  which is, even for $n = 1$, one-third of the worst-case bound 0.04 (*i.e.,* 66% less conservative).

- **Trade-off Between ϵ and $\bar{\beta}_2$.** Demanding higher accuracy (*i.e.,* smaller ϵ) reduces the confidence level $(1 - \bar{\beta}_2)$. This closed-form relation gives a lower bound for N based on ϵ and $\bar{\beta}_2$ as

$$N \geq \frac{1}{\bar{\beta}_2 \epsilon^2}\big(\operatorname{Tr}(\Gamma_\Sigma^2) + (\operatorname{Tr}(\Gamma_\Sigma))^2 + 2\lambda_{\max}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu) + 2\operatorname{Tr}(\Gamma_\Sigma)\operatorname{Tr}(\Gamma_\mu)\big).$$

- **Multiple Noise Realizations.** While N trajectories should be collected to capture the stochastic behavior of dt-SNPS, the proposed matrix inequality condition needs to be solved only once.

Discussion (cont.)

- **Infinite-Horizon Guarantees.** If noise is multiplicative *i.e.,* $\varsigma \odot x$, then $\psi = 0$. Accordingly,

$$\mathbb{P}\Big\{(x_{x_0 u} \notin X_\delta \text{ for all } k \in [0, \infty) \mid x_0\Big\} \geq 1 - \beta_1,$$

where $\beta_1 = \frac{\eta}{\delta}$.

Discussion (cont.)

- **Infinite-Horizon Guarantees.** If noise is multiplicative *i.e.,* $\varsigma \odot x$, then $\psi = 0$. Accordingly,

$$\mathbb{P}\Big\{(x_{x_0 u} \notin X_\delta \text{ for all } k \in [0, \infty) \mid x_0\Big\} \geq 1 - \beta_1,$$

  where $\beta_1 = \frac{\eta}{\delta}$.

- **Stability Analysis.** The proposed data-driven method can be naturally utilized to assess other essential system properties, such as mean-square stability.

Discussion (cont.)

- **Infinite-Horizon Guarantees.** If noise is multiplicative *i.e.,* $\varsigma \odot x$, then $\psi = 0$. Accordingly,

$$\mathbb{P}\Big\{(x_{x_0 u} \notin X_\delta \text{ for all } k \in [0, \infty) \mid x_0\Big\} \geq 1 - \beta_1,$$

  where $\beta_1 = \frac{\eta}{\delta}$.

- **Stability Analysis.** The proposed data-driven method can be naturally utilized to assess other essential system properties, such as mean-square stability.

- **Limitations.**
  - The class of nonlinear systems is restricted to polynomial dynamics.
  - S-CBC is restricted to quadratic, *i.e.* $\mathcal{B}(x) = x^\top P x$. How about higher-order polynomials $\mathcal{B}(x) = \mathcal{F}(x)^\top P \mathcal{F}(x)$?

Case Studies

System	N	T	$\mathcal{T}$	ϵ	Γ_μ	Γ_Σ	β_1	$\bar{\beta}_2$	β_2	RT1 (sec)	RT2 (sec)	MU (Mb)
Lorenz	77	10	100	0.1	$\mathbf{0}_3$	$0.006\,\mathbb{I}_3$	0.08	5×10^{-4}	0.005	0.01	1.58	13.54
Chen	328	7	100	0.2	$\mathbf{0}_3$	$0.008\,\mathbb{I}_3$	0.05	5×10^{-5}	0.00035	0.01	0.74	8.84
Spacecraft	1283	8	20	0.01	$\mathbf{0}_3$	$0.0075\,\mathbb{I}_3$	0.07	0.005	0.04	0.03	1.37	12.73

Case Studies

System	N	T	$\mathcal{T}$	ϵ	Γ_μ	Γ_Σ	β_1	$\bar{\beta}_2$	β_2	RT1 (sec)	RT2 (sec)	MU (Mb)
Lorenz	77	10	100	0.1	$\mathbf{0}_3$	$0.006\,\mathbb{I}_3$	0.08	5×10^{-4}	0.005	0.01	1.58	13.54
Chen	328	7	100	0.2	$\mathbf{0}_3$	$0.008\,\mathbb{I}_3$	0.05	5×10^{-5}	0.00035	0.01	0.74	8.84
Spacecraft	1283	8	20	0.01	$\mathbf{0}_3$	$0.0075\,\mathbb{I}_3$	0.07	0.005	0.04	0.03	1.37	12.73

- **Spacecraft System:**

$$\Upsilon: \begin{cases} x_1^+ = x_1 + \tau\,(\frac{J_2 - J_3}{J_1}\,x_2\,x_3 + \frac{1}{J_1}\,u_1) + \varsigma_1 \\ x_2^+ = x_2 + \tau\,(\frac{J_3 - J_1}{J_2}\,x_1\,x_3 + \frac{1}{J_2}\,u_2) + \varsigma_2 \\ x_3^+ = x_3 + \tau\,(\frac{J_1 - J_2}{J_3}\,x_1\,x_2 + \frac{1}{J_3}\,u_3) + \varsigma_3 \end{cases}$$

- **Regions of Interest:** $X = [-10, 10]^3$, $X_\eta = [-1, 1]^3$, $X_\delta = [-10, -6]^3 \cup [6, 10]^3$

Case Studies

System	N	T	$\mathcal{T}$	ϵ	Γ_μ	Γ_Σ	β_1	$\bar{\beta}_2$	β_2	RT1 (sec)	RT2 (sec)	MU (Mb)
Lorenz	77	10	100	0.1	$\mathbf{0}_3$	$0.006\,\mathbb{I}_3$	0.08	5×10^{-4}	0.005	0.01	1.58	13.54
Chen	328	7	100	0.2	$\mathbf{0}_3$	$0.008\,\mathbb{I}_3$	0.05	5×10^{-5}	0.00035	0.01	0.74	8.84
Spacecraft	1283	8	20	0.01	$\mathbf{0}_3$	$0.0075\,\mathbb{I}_3$	0.07	0.005	0.04	0.03	1.37	12.73

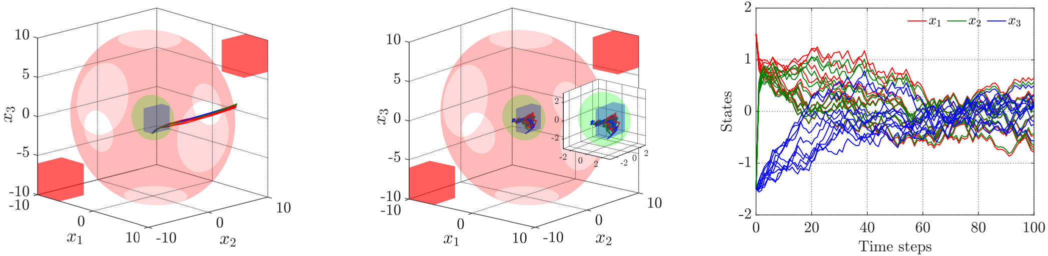
- **Spacecraft System:**

$$\Upsilon: \begin{cases} x_1^+ = x_1 + \tau\,(\frac{J_2-J_3}{J_1}\,x_2\,x_3 + \frac{1}{J_1}\,u_1) + \varsigma_1 \\ x_2^+ = x_2 + \tau\,(\frac{J_3-J_1}{J_2}\,x_1\,x_3 + \frac{1}{J_2}\,u_2) + \varsigma_2 \\ x_3^+ = x_3 + \tau\,(\frac{J_1-J_2}{J_3}\,x_1\,x_2 + \frac{1}{J_3}\,u_3) + \varsigma_3 \end{cases}$$

- $Z = [-0.15, 0.15]^3$, accordingly $\varpi^2 = 0.0675$
- $\Gamma_\Sigma + \Gamma_\mu + \epsilon\mathbb{I}_3 = 0.0175\mathbb{I}_3$ (74% smaller in each entry)

- **Regions of Interest:** $X = [-10, 10]^3$, $X_\eta = [-1, 1]^3$, $X_\delta = [-10, -6]^3 \cup [6, 10]^3$

Simulation Results: Lorenz System



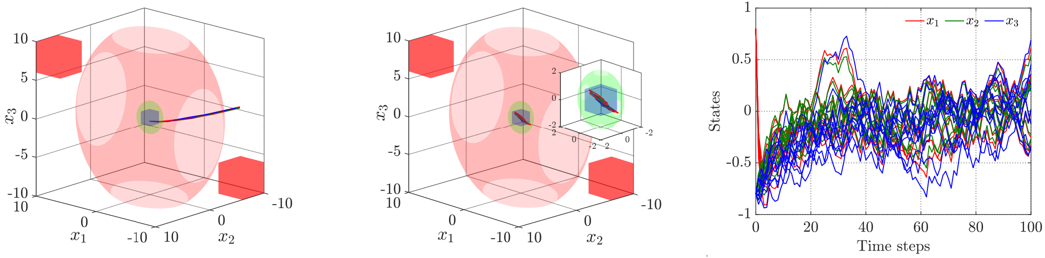
- **Designed S-CBC Matrix P:**

$$P = \begin{bmatrix} 33450.3 & 3339.99 & 656.04 \\ 3339.99 & 38333 & 1417.8 \\ 656.04 & 1417.8 & 34290.3 \end{bmatrix}$$

- **Designed Controller:**

$$u = -1.4389\,x_1^2 - 1.4844\,x_1x_2 - 0.50617\,x_1x_3 + 1.7463\,x_2^2 + 0.62559\,x_2x_3 + 0.1802\,x_3^2 \\ - 1.9406\,x_1 - 28.1245\,x_2 + 0.1565\,x_3$$

Simulation Results: Chen System



- **Designed S-CBC Matrix P:**

$$P = \begin{bmatrix} 65587.8 & 20628.6 & 486.86 \\ 20628.6 & 103532 & 984.94 \\ 486.86 & 984.94 & 60375.3 \end{bmatrix}$$

- **Designed Controller:**

$$u = 0.016422\, x_1^2 - 0.12604\, x_1 x_2 - 0.48379\, x_1 x_3 + 0.0019506\, x_2^2 + 2.2790\, x_2 x_3 + 0.030116\, x_3^2 + 4.9195\, x_1 - 32.3368\, x_2 + 0.38473\, x_3$$

Core Insights

- When robust analysis is feasible, it can be more advantageous, as it ensures safety without introducing any risk to the system.
- Otherwise, despite introducing some probabilistic risk, stochastic analysis can be less conservative than robust analysis.

Core Insights

- When robust analysis is feasible, it can be more advantageous, as it ensures safety without introducing any risk to the system.
- Otherwise, despite introducing some probabilistic risk, stochastic analysis can be less conservative than robust analysis.

Future work: Develop physics-informed data-driven techniques[3] that can significantly reduce the trajectory horizon

[3] M.H. Ashoori, A. Aminzadeh, A. Nejati, and A. Lavaei, “Physics-Informed Data-Driven Control of Nonlinear Polynomial Systems with Noisy Data", *IEEE TAC*, under review, 2025.

Core Insights

- When robust analysis is feasible, it can be more advantageous, as it ensures safety without introducing any risk to the system.
- Otherwise, despite introducing some probabilistic risk, stochastic analysis can be less conservative than robust analysis.

Future work: Develop physics-informed data-driven techniques[3] that can significantly reduce the trajectory horizon

Thank you for your attention!

[3] M.H. Ashoori, A. Aminzadeh, A. Nejati, and A. Lavaei, "Physics-Informed Data-Driven Control of Nonlinear Polynomial Systems with Noisy Data", *IEEE TAC*, under review, 2025.