

Data-Driven Stochastic Control with Probabilistic Guarantees

Abolfazl Lavaei

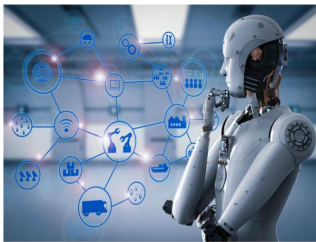
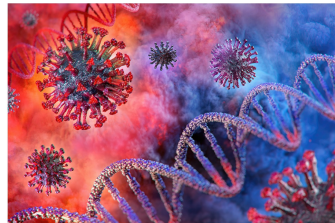
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Stochastic Control Systems



Stochastic Control: Challenges



- **Potential difficulties:**

- Stochastic nature of dynamics

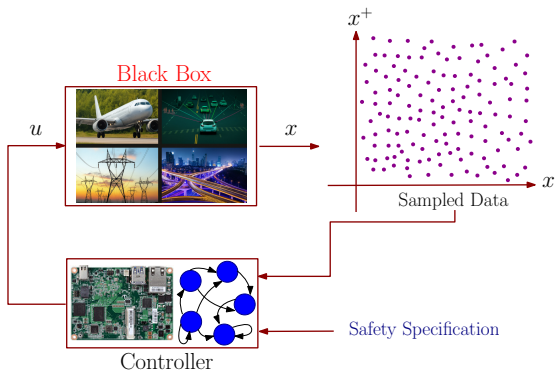
Stochastic Control: Challenges



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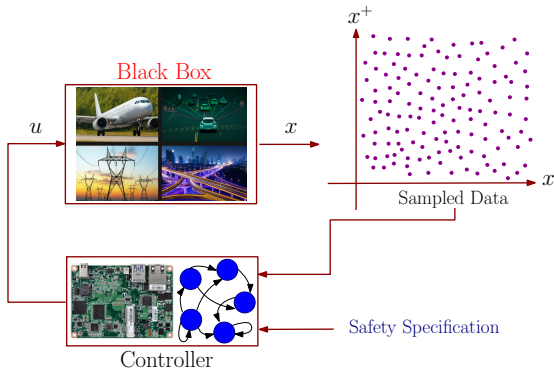
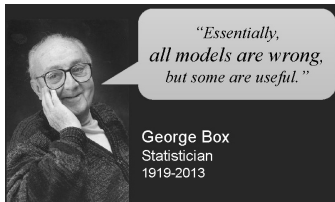
- Stochastic nature of dynamics
- Lack of mathematical models

Data-Driven Analysis with Provable Guarantees

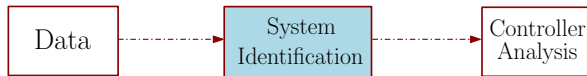


Data-Driven Analysis with Provable Guarantees

- Closed-form models: not available or too complex to deal with
- Model-based techniques cannot be useful

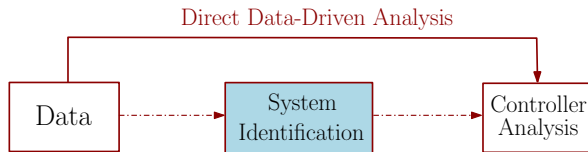


Data-Driven Control: Indirect vs. Direct Methods



- Indirect data-driven techniques: System identification followed by model-based methods
 - Two-level computational complexity

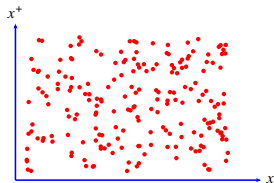
Data-Driven Control: Indirect vs. Direct Methods



- Indirect data-driven techniques: System identification followed by model-based methods
 - Two-level computational complexity
- Direct data-driven techniques: Directly employ system measurements
 - More samples for robustness guarantees

Direct Data-Driven Control: Scenario-based vs. Trajectory-based Approaches

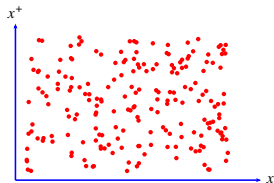
Scenario-based Approach



● i.i.d. samples

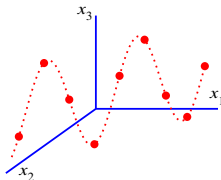
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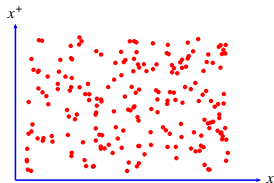
Trajectory-based Approach



- One set of (**non**-i.i.d.) time-series data

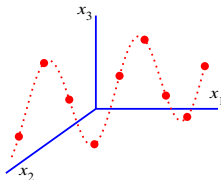
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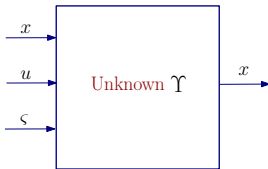


- One set of (**non**-i.i.d.) time-series data

Question of interest

How to design safety controllers with **probabilistic confidence** using **trajectory-based** approaches for **stochastic** control systems with **unknown models**?

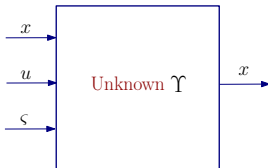
Discrete-time Nonlinear Polynomial Systems (dt-NPS)



$$\Upsilon: x^+ = f(x) + g(x)u + \varsigma$$

- $x \in X \subseteq \mathbb{R}^n$ and $u \in U = \mathbb{R}^m$
- $f: X \rightarrow X$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$
- $\varsigma \in Z \subseteq \mathbb{R}^n$, where $\|\varsigma\| \leq \varpi \in \mathbb{R}_0^+$

Discrete-time Nonlinear Polynomial Systems (dt-NPS)



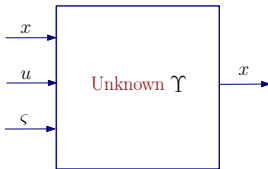
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Robust Control Barrier Certificates (R-CBC) & Robust Safety Controller (R-SC)

R-CBC and R-SC¹

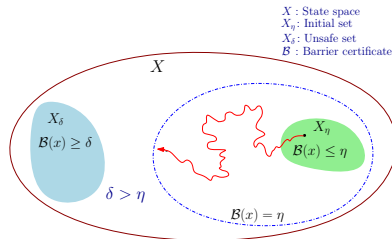
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• $\forall x \in X_\eta: \mathcal{B}(x) \leq \eta,$

• $\forall x \in X_\delta: \mathcal{B}(x) \geq \delta,$

• $\forall x \in X, \exists u \in U$, such that $\forall \varsigma \in Z:$

$$\mathcal{B}(A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma) \leq \kappa\mathcal{B}(x) + \rho\|\varsigma\|^2$$



[1] O. Akbarzadeh, M.H. Ashoori, and A. Lavaei, "Learning Robust Safety Controllers for Uncertain Input-Affine Polynomial Systems", *CDC 2025*.

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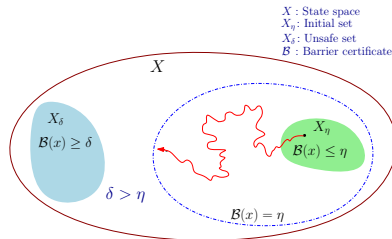
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If

$$\rho\varpi \leq \delta(1 - \kappa),$$

then $x_{x_0 u w}(k) \notin X_\delta$ for any $x_0 \in X_0$ and $k \in \mathbb{N}$ under signals $u(\cdot)$ and $\varsigma(\cdot)$.



[1] O. Akbarzadeh, M.H. Ashoori, and A. Lavaei, "Learning Robust Safety Controllers for Uncertain Input-Affine Polynomial Systems", CDC 2025.

Data-Driven Design of R-CBC and R-SC

- $\vec{x} = [x(1) \quad x(2) \quad \dots \quad x(T)]$
- $\mathbb{X} = [x(0) \quad x(1) \quad \dots \quad x(T-1)]$
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$$\vec{\mathbb{X}}_j = A\mathbb{F}_j + B\mathbb{G}_j + \mathbb{Z}_j = \Phi\mathbb{H}_j + \mathbb{Z}_j, \quad j \in \{1, \dots, T\}$$

▷ $\mathbb{Z}_j = \vec{\mathbb{X}}_j - \Phi\mathbb{H}_j$

where $\Phi = [A \quad B], \quad \mathbb{H}_j = \begin{bmatrix} \mathbb{F}_j \\ \mathbb{G}_j \end{bmatrix}$

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▷ $\mathbb{Z}_j = \vec{\mathbb{X}}_j - \Phi\mathbb{H}_j$

$$\mathbb{Z}_j \mathbb{Z}_j^\top \preceq \varpi^2 \mathbb{I}_n$$

▷ $(\vec{\mathbb{X}}_j - \Phi\mathbb{H}_j)(\vec{\mathbb{X}}_j - \Phi\mathbb{H}_j)^\top - \varpi^2 \mathbb{I}_n \preceq 0 \quad \rightarrow \quad \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix}^\top \begin{bmatrix} \vec{\mathbb{X}}_j \vec{\mathbb{X}}_j^\top - \varpi^2 \mathbb{I}_n & -\vec{\mathbb{X}}_j \mathbb{H}_j^\top \\ * & \mathbb{H}_j \mathbb{H}_j^\top \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix} \preceq 0$

Data-Driven Design of R-CBC and R-SC: Main Result

Main Result: R-CBC and R-SC Design

Given an unknown dt-NPS Υ , let there exist $\bar{\eta}, \bar{\delta} \in \mathbb{R}^+$, with $\bar{\eta} > \bar{\delta}$, $\kappa \in (0, 1)$, matrices $\bar{P} \succ 0$ and $\bar{K}(x)$, and $\alpha_{j=1, \dots, T}(x): \mathbb{R}^n \rightarrow \mathbb{R}_0^+$, such that

$$\bullet \quad \bar{P} - \bar{\eta} z_{\eta} z_{\eta}^{\top} \succeq 0, \quad \forall x \in X_{\eta}, \quad \text{with } X_{\eta} \subseteq \{x \in \mathbb{R}^n: xx^{\top} \preceq z_{\eta} z_{\eta}^{\top}, z_{\eta} \in \mathbb{R}^{n \times n}\},$$

$$\bullet \quad \bar{P} - \bar{\delta} z_{\delta} z_{\delta}^{\top} \preceq 0, \quad \forall x \in X_{\delta}, \quad \text{with } X_{\delta} \subseteq \{x \in \mathbb{R}^n: xx^{\top} \succeq z_{\delta} z_{\delta}^{\top}, z_{\delta} \in \mathbb{R}^{n \times n}\},$$

$$\bullet \quad \begin{bmatrix} -\kappa \bar{P} & 0 \\ * & 0 \\ * & * \end{bmatrix} \begin{bmatrix} 0 \\ \mathcal{J}(x) \bar{P} \\ \mathcal{G}(x) \bar{K}(x) \\ -(1 + \mu)^{-1} \bar{P} \end{bmatrix} - \sum_{j=1}^T \alpha_j(x) \begin{bmatrix} \mathcal{R}^{DC_j} & 0 \\ * & 0 \end{bmatrix} \preceq 0, \quad \forall x \in X,$$

with

$$\mathcal{R}^{DC_j} = \begin{bmatrix} \vec{\mathbb{X}}_j \vec{\mathbb{X}}_j^{\top} - \varpi^2 \mathbb{I}_n & -\vec{\mathbb{X}}_j \mathbb{H}_j^{\top} \\ * & \mathbb{H}_j \mathbb{H}_j^{\top} \end{bmatrix}.$$

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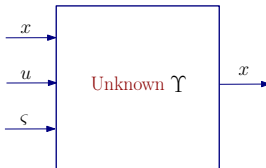
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Then, $\mathcal{B}(x) = x^\top P x$, with $P = \bar{P}^{-1}$, and $u = \mathcal{K}(x)x$, with $\mathcal{K}(x) = \bar{\mathcal{K}}(x) \bar{P}^{-1} = \bar{\mathcal{K}}(x) P$. In addition, $\eta = \bar{\eta}^{-1}$, $\delta = \bar{\delta}^{-1}$ (where $\eta < \delta$), and $\rho = (1 + \mu^{-1}) \|\sqrt{P}\|^2$.

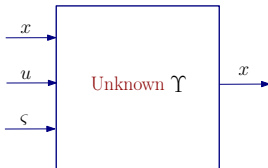
Discrete-time **Stochastic** Nonlinear Polynomial Systems (dt-SNPS)



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- $\mathcal{G}(x) \in \mathbb{R}^{q \times m}$
- $(\Omega, \mathbb{F}_\Omega, \mathbb{P}_\Omega)$
- $\varsigma := \{\varsigma(k) : \Omega \rightarrow \mathcal{V}_\varsigma, k \in \mathbb{N}\};$
- $\mu\mu^\top \preceq \Gamma_\mu$ and $\Sigma \preceq \Gamma_\Sigma$

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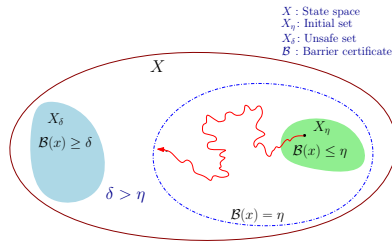
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Stochastic Control Barrier Certificates (S-CBC) & Safety Controller

S-CBC and Safety Controller²

Consider a dt-NPS Υ with $X_\eta, X_\delta \subseteq X$. Suppose there exist $\mathcal{B} : X \rightarrow \mathbb{R}_0^+$ and $\eta, \delta, \psi \in \mathbb{R}_0^+$, with $\delta > \eta$, and $\kappa \in (0, 1)$, such that:

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- $\forall x \in X, \exists u \in U$, such that:
$$\mathbb{E} \left[\mathcal{B}(A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma) \mid x, u \right] \leq \kappa \mathcal{B}(x) + \psi.$$



[2] A. Lavaei, "Data-Driven Stochastic Control: Foundations and Guarantees", *IEEE TAC*, under revision, 2025.

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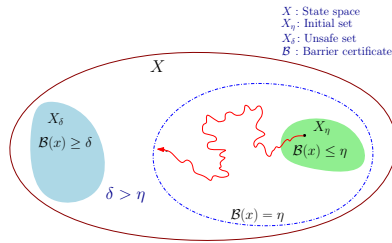
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$$\mathbb{E} \left[\mathcal{B}(A\mathcal{F}(x) + B\mathcal{G}(x)u + \varsigma) \mid x, u \right] \leq \kappa \mathcal{B}(x) + \psi.$$

Then:

$$\mathbb{P} \left\{ x_{x_0 u} \notin X_\delta \text{ for all } k \in [0, \mathcal{T}] \mid x_0 \right\} \geq 1 - \beta_1,$$

where $\beta_1 = \frac{\eta + \psi \mathcal{T}}{\delta}.$



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Data-Driven Design of S-CBC and Safety Controller

For all $i \in \{1, 2, \dots, N\}$:

$$\vec{\mathcal{X}}^i = [x^i(1) \quad x^i(2) \quad \dots \quad x^i(T)]$$

$$\mathcal{X}^i = [x(0) \quad x^i(1) \quad \dots \quad x^i(T-1)]$$

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$$\mathcal{Z}^i = [\varsigma^i(0) \quad \varsigma^i(1) \quad \dots \quad \varsigma^i(T-1)]$$

Data-Driven Design of S-CBC and Safety Controller

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- $\vec{x}^i = [x^i(1) \quad x^i(2) \quad \dots \quad x^i(T)]$

- $\mathbb{X}^i = [x(0) \quad x^i(1) \quad \dots \quad x^i(T-1)]$

- $\mathbb{U} = [u(0) \quad u(1) \quad \dots \quad u(T-1)]$

- $\mathbb{Z}^i = [\varsigma^i(0) \quad \varsigma^i(1) \quad \dots \quad \varsigma^i(T-1)]$

- $\mathbb{F}^i = [\mathcal{F}(x(0)) \quad \mathcal{F}(x^i(1)) \quad \dots \quad \mathcal{F}(x^i(T-1))]$

- $\mathbb{G}^i = [\mathcal{G}(x(0))u(0) \quad \mathcal{G}(x^i(1))u(1) \quad \dots \quad \mathcal{G}(x^i(T-1))u(T-1)]$

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Problem of interest

Given an unknown dt-SNPS with process noise ς , design an **S-CBC and its safety controller**, while quantifying the **probabilistic safety level** $\beta_1 \in (0, 1]$ and **confidence level** $\beta_2 \in (0, 1]$ purely based on data, *i.e.*,

$$\mathbb{P}\left\{\mathbb{P}\{\Upsilon \models \mathbb{S}\} \geq 1 - \beta_1\right\} \geq 1 - \beta_2.$$

Data-Driven Design of S-CBC and Safety Controller

$$\mathbb{E} \left[\varsigma \varsigma^\top \right] = \Sigma + \mu \mu^\top \preceq \Gamma_\Sigma + \Gamma_\mu$$

Data-Driven Design of S-CBC and Safety Controller

$$\mathbb{E}[\zeta\zeta^\top] = \Sigma + \mu\mu^\top \preceq \Gamma_\Sigma + \Gamma_\mu$$

Lemma 1: Formalizing Empirical Approximation

Let $\mathbb{Z}_{(\cdot)}^1, \dots, \mathbb{Z}_{(\cdot)}^N \in \mathbb{R}^n$ be N independent samples drawn from $\zeta \in \mathbb{R}^n$, where $\mu\mu^\top \preceq \Gamma_\mu$ and $\Sigma \preceq \Gamma_\Sigma$. Then

$$\mathbb{P}\left(\left\|\frac{1}{N} \sum_{i=1}^N \mathbb{Z}_{(\cdot)}^i \mathbb{Z}_{(\cdot)}^{i\top} - \mathbb{E}[\zeta\zeta^\top]\right\| < \epsilon\right) \geq 1 - \bar{\beta}_2,$$

with $\bar{\beta}_2 = \frac{1}{N\epsilon^2} (\text{Tr}(\Gamma_\Sigma^2) + (\text{Tr}(\Gamma_\Sigma))^2 + 2\lambda_{\max}(\Gamma_\Sigma) \text{Tr}(\Gamma_\mu) + 2 \text{Tr}(\Gamma_\Sigma) \text{Tr}(\Gamma_\mu))$.

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Lemma 2: Data-Conformity Constraint

Under [Lemma 1](#), the following statement holds true with a confidence of at least $1 - \bar{\beta}_2$:

$$\frac{1}{N} \sum_{i=1}^N \mathbb{Z}_{(\cdot)}^i \mathbb{Z}_{(\cdot)}^{i\top} \preceq \Gamma_\Sigma + \Gamma_\mu + \epsilon \mathbb{I}_n$$

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$$\rightarrow \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix}^\top \left[\frac{1}{N} \sum_{i=1}^N \underbrace{\overrightarrow{\mathbb{X}}_j^i \overrightarrow{\mathbb{X}}_j^{i\top}}_* - (\Gamma_\Sigma + \Gamma_\mu + \epsilon \mathbb{I}_n) - \frac{1}{N} \sum_{i=1}^N \overrightarrow{\mathbb{X}}_j^i \mathbb{H}_j^{i\top} \right] \begin{bmatrix} \mathbb{I}_n \\ \Phi^\top \end{bmatrix} \preceq 0$$

Data-Driven Design of S-CBC and Safety Controller: Main Result

Main Result: S-CBC and Safety Controller Design

Given an unknown dt-NPS Υ , let there exist $\bar{\eta}, \bar{\delta} \in \mathbb{R}^+$, with $\bar{\eta} > \bar{\delta}$, $\kappa \in (0, 1)$, matrices $\bar{P} \succ 0$ and $\bar{\mathcal{K}}(x)$, and $\alpha_{j=1, \dots, T}(x): \mathbb{R}^n \rightarrow \mathbb{R}_0^+$, such that

- $\bar{P} - \bar{\eta} z_\eta z_\eta^\top \succeq 0, \quad \forall x \in X_\eta, \quad \text{with } X_\eta \subseteq \{x \in \mathbb{R}^n: xx^\top \preceq z_\eta z_\eta^\top, z_\eta \in \mathbb{R}^{n \times n}\},$

- $\bar{P} - \bar{\delta} z_\delta z_\delta^\top \preceq 0, \quad \forall x \in X_\delta, \quad \text{with } X_\delta \subseteq \{x \in \mathbb{R}^n: xx^\top \succeq z_\delta z_\delta^\top, z_\delta \in \mathbb{R}^{n \times n}\},$

- $$\begin{bmatrix} -\kappa \bar{P} & 0 & 0 \\ * & 0 & \begin{bmatrix} \mathcal{J}(x) \bar{P} \\ \mathcal{G}(x) \bar{\mathcal{K}}(x) \end{bmatrix} \\ * & * & -(1 + \rho)^{-1} \bar{P} \end{bmatrix} - \sum_{j=1}^T \alpha_j(x) \begin{bmatrix} \mathcal{R}^{DC_j} & 0 \\ * & 0 \end{bmatrix} \preceq 0, \quad \forall x \in X,$$

with

$$\mathcal{R}^{DC_j} = \begin{bmatrix} \frac{1}{N} \sum_{i=1}^N \bar{\mathcal{X}}_j^i \bar{\mathcal{X}}_j^{i\top} - (\Gamma_\Sigma + \Gamma_\mu + \epsilon \mathbb{I}_n) & -\frac{1}{N} \sum_{i=1}^N \bar{\mathcal{X}}_j^i \mathbb{H}_j^{i\top} \\ * & \frac{1}{N} \sum_{i=1}^N \mathbb{H}_j^i \mathbb{H}_j^{i\top} \end{bmatrix}, \quad \text{where } \mathbb{H}_j^i = \begin{bmatrix} \mathbb{F}_j^i \\ \mathbb{G}_j^i \end{bmatrix}.$$

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$$\bullet \quad \bar{P} - \bar{\eta} z_{\eta} z_{\eta}^{\top} \succeq 0, \quad \forall x \in X_{\eta}, \quad \text{with } X_{\eta} \subseteq \{x \in \mathbb{R}^n: xx^{\top} \preceq z_{\eta} z_{\eta}^{\top}, z_{\eta} \in \mathbb{R}^{n \times n}\},$$

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Then, $\mathcal{B}(x) = x^{\top} P x$, with $P = \bar{P}^{-1}$, and $u = \mathcal{K}(x)x$, with $\mathcal{K}(x) = \bar{\mathcal{K}}(x) \bar{P}^{-1}$. In addition, $\eta = \bar{\eta}^{-1}$, $\delta = \bar{\delta}^{-1}$ ($\eta < \delta$), and $\psi = (1 + \rho^{-1}) \text{Tr}(P \Gamma_{\mu}) + \text{Tr}(P \Gamma_{\Sigma})$, with a confidence of at least $1 - \beta_2$, where $\beta_2 = T \bar{\beta}_2$.

Require: The state set X , extended dictionaries $\mathcal{F}(x), \mathcal{G}(x)$, and bounds $\Gamma_\mu, \Gamma_\Sigma$

- 1: Collect $\vec{X}^i, \mathbb{X}^i$, and $\mathbb{U}$, where $i \in \{1, 2, \dots, N\}$
- 2: Construct $\mathbb{F}^i$ and $\mathbb{G}^i$
- 3: Initialize the desired ϵ and N , and compute $\bar{\beta}_2$
- 4: Initialize $\kappa \in (0, 1)$ and $\rho \in \mathbb{R}^+$
- 5: Solve conditions using SeDuMi and SOSTOOLS and compute $P = \bar{P}^{-1}$, $\mathcal{K}(x) = \bar{\mathcal{K}}(x)\bar{P}^{-1} = \bar{\mathcal{K}}(x)P$, $\eta = \bar{\eta}^{-1}$, and $\delta = \bar{\delta}^{-1}$
- 6: Compute ψ
- 7: Quantify β_1 and $\beta_2 = T\bar{\beta}_2$

Ensure: S-CBC $\mathcal{B}(x) = x^\top P x$, safety controller $u = \mathcal{K}(x)x$, and guaranteed probabilistic safety

$$\mathbb{P}\left\{\mathbb{P}\{\Upsilon \models \mathbb{S}\} \geq 1 - \beta_1\right\} \geq 1 - \beta_2$$

Discussion

- **Robust vs. Stochastic Analysis.** If $\varsigma \in Z = [-0.2, 0.2]^n$, then $\varpi^2 = 0.04n$ (yielding $0.04n\mathbb{I}_n$ in robust condition). In contrast, if $\varsigma \sim \mathcal{U}(\underbrace{-0.2}_a, \underbrace{0.2}_b)^n$, then

$$\mathbb{E}[\varsigma\varsigma^\top] = \Sigma + \mu\mu^\top = \frac{(b-a)^2}{12}\mathbb{I}_n + \left(\frac{a+b}{2}\right)^2\mathbf{1}_n\mathbf{1}_n^\top = \frac{0.16}{12}\mathbb{I}_n \approx 0.0133\mathbb{I}_n,$$

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- **Multiple Noise Realizations.** While N trajectories should be collected to capture the stochastic behavior of dt-SNPS, the proposed matrix inequality condition needs to be solved **only once**.
- **Limitations.**
 - The class of nonlinear systems is restricted to **polynomial dynamics**.
 - S-CBC is restricted to **quadratic**, *i.e.* $\mathcal{B}(x) = x^\top Px$. How about higher-order polynomials $\mathcal{B}(x) = \mathcal{F}(x)^\top P\mathcal{F}(x)$?

Case Studies

System	N	T	$\mathcal{T}$	ϵ	Γ_μ	Γ_Σ	β_1	$\bar{\beta}_2$	β_2	RT1 (sec)	RT2 (sec)	MU (Mb)
Lorenz	77	10	100	0.1	$\mathbf{0}_3$	$0.006 \mathbb{I}_3$	0.08	5×10^{-4}	0.005	0.01	1.58	13.54
Chen	328	7	100	0.2	$\mathbf{0}_3$	$0.008 \mathbb{I}_3$	0.05	5×10^{-5}	0.00035	0.01	0.74	8.84
Spacecraft	1283	8	20	0.01	$\mathbf{0}_3$	$0.0075 \mathbb{I}_3$	0.07	0.005	0.04	0.03	1.37	12.73

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Spacecraft System:

$$\Upsilon: \begin{cases} x_1^+ = x_1 + \tau \left(\frac{J_2 - J_3}{J_1} x_2 x_3 + \frac{1}{J_1} u_1 \right) + \varsigma_1 \\ x_2^+ = x_2 + \tau \left(\frac{J_3 - J_1}{J_2} x_1 x_3 + \frac{1}{J_2} u_2 \right) + \varsigma_2 \\ x_3^+ = x_3 + \tau \left(\frac{J_1 - J_2}{J_3} x_1 x_2 + \frac{1}{J_3} u_3 \right) + \varsigma_3 \end{cases}$$

Regions of Interest: $X = [-10, 10]^3$, $X_\eta = [-1, 1]^3$, $X_\delta = [-10, -6]^3 \cup [6, 10]^3$

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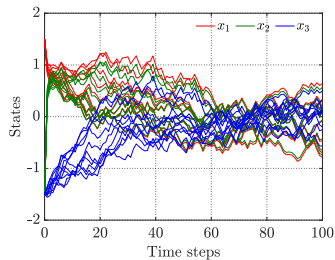
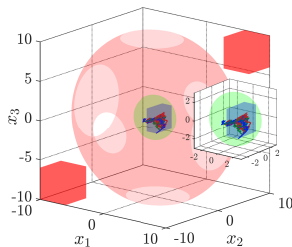
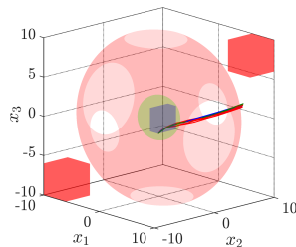
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- $Z = [-0.15, 0.15]^3$, accordingly $\varpi^2 = 0.0675$
- $\Gamma_\Sigma + \Gamma_\mu + \epsilon \mathbb{I}_3 = 0.0175 \mathbb{I}_3$ (74% smaller in each entry)

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Simulation Results: Lorenz System



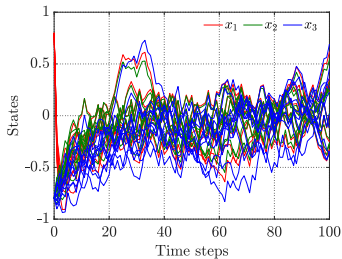
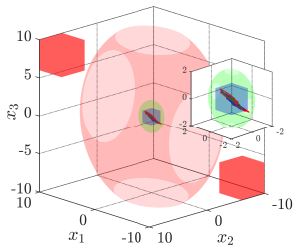
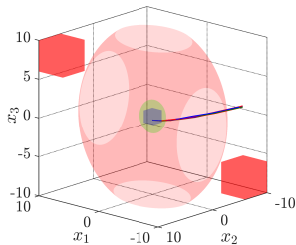
Designed S-CBC Matrix P :

$$P = \begin{bmatrix} 33450.3 & 3339.99 & 656.04 \\ \star & 38333 & 1417.8 \\ \star & \star & 34290.3 \end{bmatrix}$$

Designed Controller:

$$u = -1.4389 x_1^2 - 1.4844 x_1 x_2 - 0.50617 x_1 x_3 + 1.7463 x_2^2 + 0.62559 x_2 x_3 + 0.1802 x_3^2 \\ - 1.9406 x_1 - 28.1245 x_2 + 0.1565 x_3$$

Simulation Results: Chen System



Designed S-CBC Matrix P :

$$P = \begin{bmatrix} 65587.8 & 20628.6 & 486.86 \\ \star & 103532 & 984.94 \\ \star & \star & 60375.3 \end{bmatrix}$$

Designed Controller:

$$u = 0.016422 x_1^2 - 0.12604 x_1 x_2 - 0.48379 x_1 x_3 + 0.0019506 x_2^2 + 2.2790 x_2 x_3 + 0.030116 x_3^2 \\ + 4.9195 x_1 - 32.3368 x_2 + 0.38473 x_3$$

Core Insights

- When robust analysis is feasible, it can be more advantageous, as it ensures safety **without introducing any risk** to the system.
- Otherwise, despite introducing some probabilistic risk, stochastic analysis can be **less conservative** than robust analysis.

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Future work: Develop **physics-informed** data-driven techniques³ that can **significantly reduce** the trajectory horizon

[3] M.H. Ashoori, A. Aminzadeh, A. Nejati, and A. Lavaei, "Physics-Informed Data-Driven Control of Nonlinear Polynomial Systems with Noisy Data", *IEEE TAC*, under revision, 2025.

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Thank you for your attention!



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