Design with Formal Risk Guarantees via the Scenario Approach: A Sample Compression Framework

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Many thanks to all collaborators!

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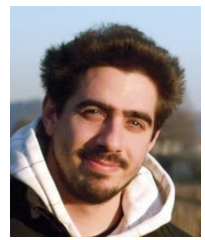
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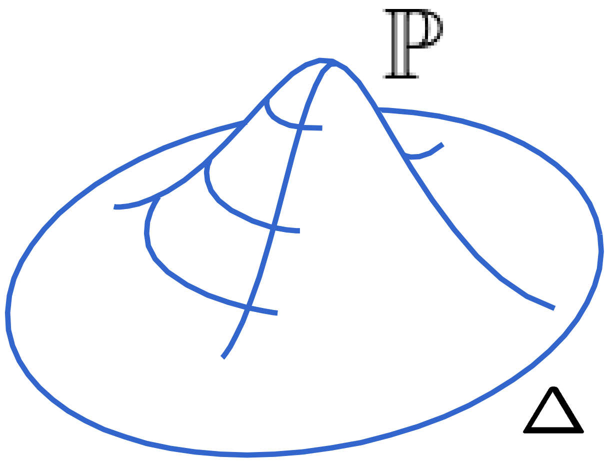
Data-driven decision-making

δ = uncertain element ⇒ exercise caution



decision $\mathcal{H}$

Data-driven decision-making

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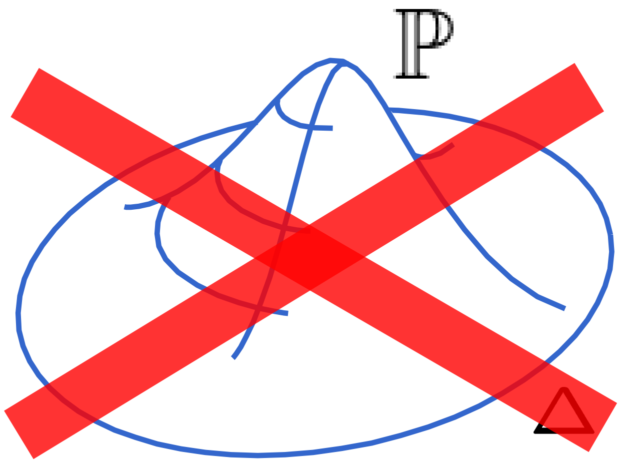

⇨

⇨ **decision** $\mathcal{H}$

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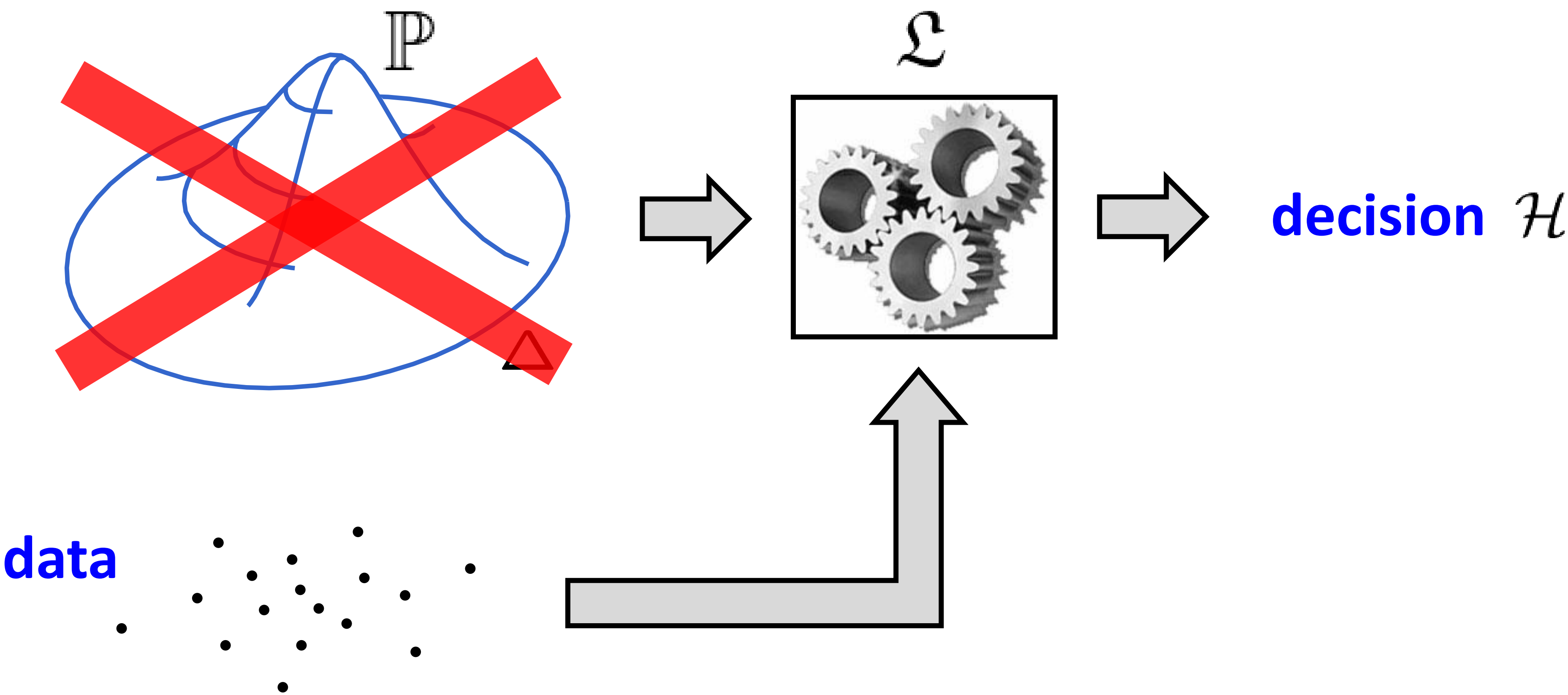


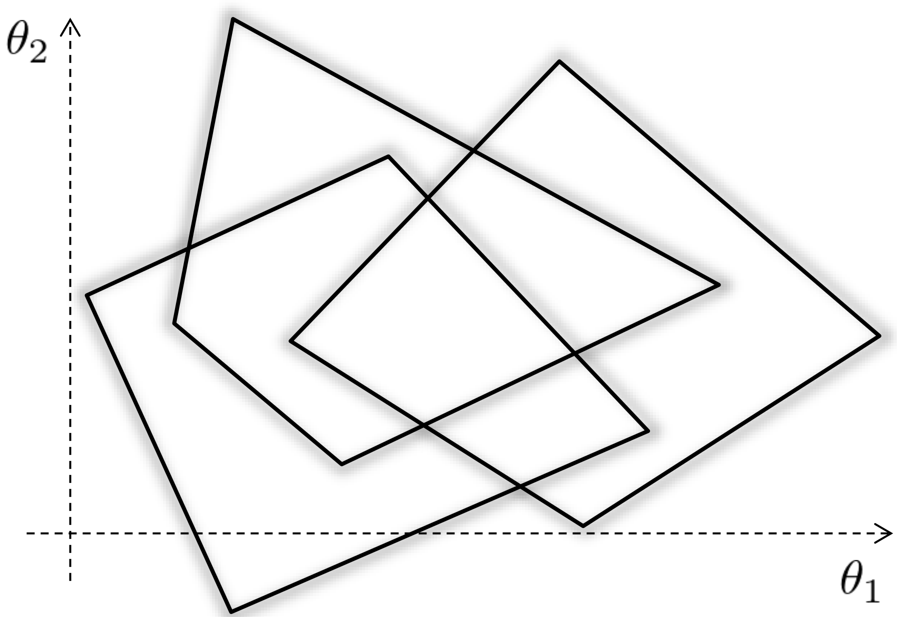
$\delta_1, \delta_2, \ldots, \delta_N$ (scenarios, i.i.d. sample)

Example: data-driven (scenario) robust optimization

data

$$\delta_i \rightarrow f(\theta, \delta_i) \leq 0$$

constraint



Example: data-driven (scenario) robust optimization

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constraint

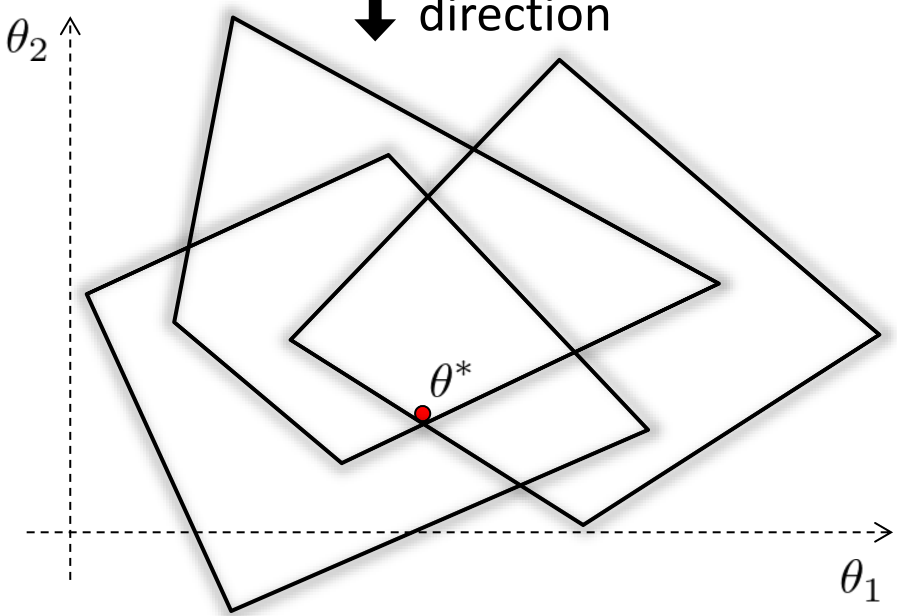
$\mathcal{H}$ = θ^* = solution to

$$\min_{\theta \in \Theta} \quad c(\theta)$$

$$\text{s.t.} \quad f(\theta, \delta_i) \leq 0$$

$$i = 1, \ldots, N$$

optimization direction



data-driven
robust H_2 control

Example: optimization with constraints relaxations

data

$$\delta_i \rightarrow f(\theta, \delta_i) \leq 0$$

constraint

$\mathcal{H}$ = θ^* = solution to

$$\min_{\theta \in \Theta, \xi_i \geq 0} \quad c(\theta) + \rho \sum_{i=1}^{N} \xi_i$$

$$\text{s.t.} \quad f(\theta, \delta_i) \leq \xi_i$$

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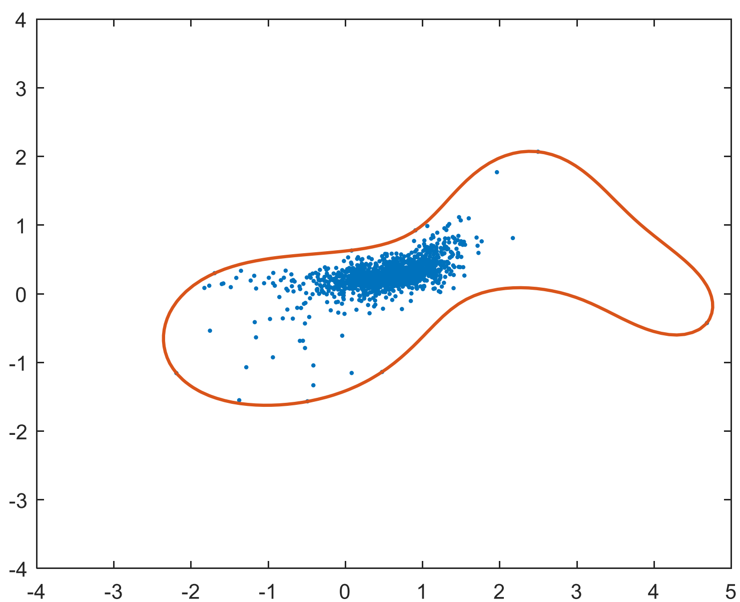


Support Vector methods
for reachability analysis

Example: classification

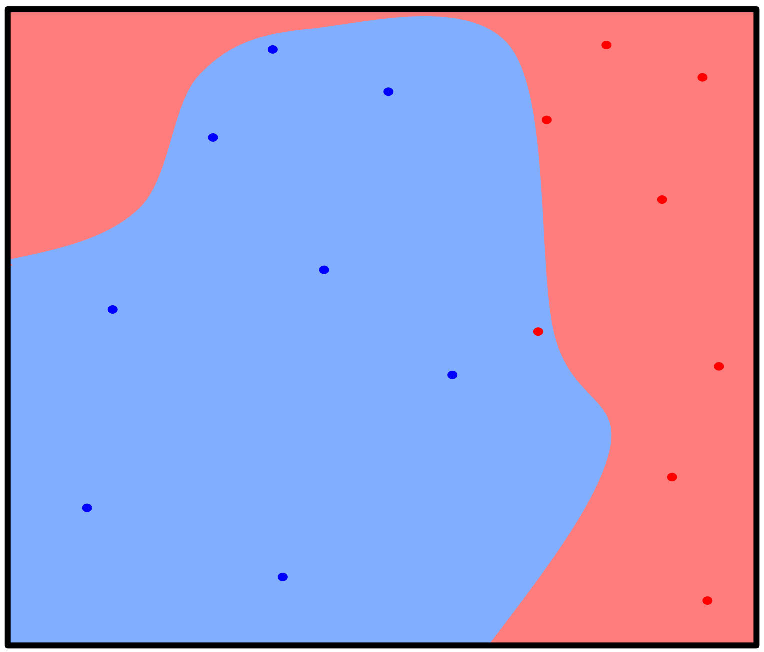
data

$\delta_i = (u_i, y_i)$

$u_i \in \mathbb{R}^d$

$y_i \in \{\text{red}, \text{blue}\}$

$\mathcal{H}$ = Neural Network classifier trained via an SGD-based training algorithm



A lesson from machine learning

Which is the data-driven decision scheme for the problem at hand?

Difficult to say a-priori without incurring in over-conservatism … a blend of approximate knowledge and heuristics, often in various attempts (hyperparameters tuning)

No limits in exploration, but some guidance is needed…

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Which is the data-driven decision scheme for the problem at hand?

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No limits in exploration, but some guidance is needed…

⇒ **SCENARIO APPROACH**: a tool to provide accurate and rigorous certification of the actual performance of the explored decisions

→ final decision selection

→ dependable utilization of it

… when is it possible?

The risk of a decision

$\mathcal{H}$ is inappropriate for a new δ

↓

interaction between decision and environment

E.g.,

a new constraint is violated by $\mathcal{H}$

a new terminal state is outside $\mathcal{H}$

a new I/O pair is misclassified by $\mathcal{H}$

The risk of a decision

$$\mathrm{R}(\mathcal{H}) = \mathbb{P}\left\{ \mathcal{H} \text{ is inappropriate for a new } \delta \right\}$$

Risk = out-of-sample probability of inappropriateness

E.g., $\mathbb{P}\{\text{a new constraint is violated by } \mathcal{H}\}$

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$\mathbb{P}$? $\{$ a new terminal state is outside $\mathcal{H}\}$

$\mathbb{P}$? $\{$a new I/O pair is misclassified by $\mathcal{H}\}$

Issue: $\mathbb{P}$ is **not** available…

Main goal

⇨ **assess $\mathrm{R}(\mathcal{H})$ from data, the same used for design**

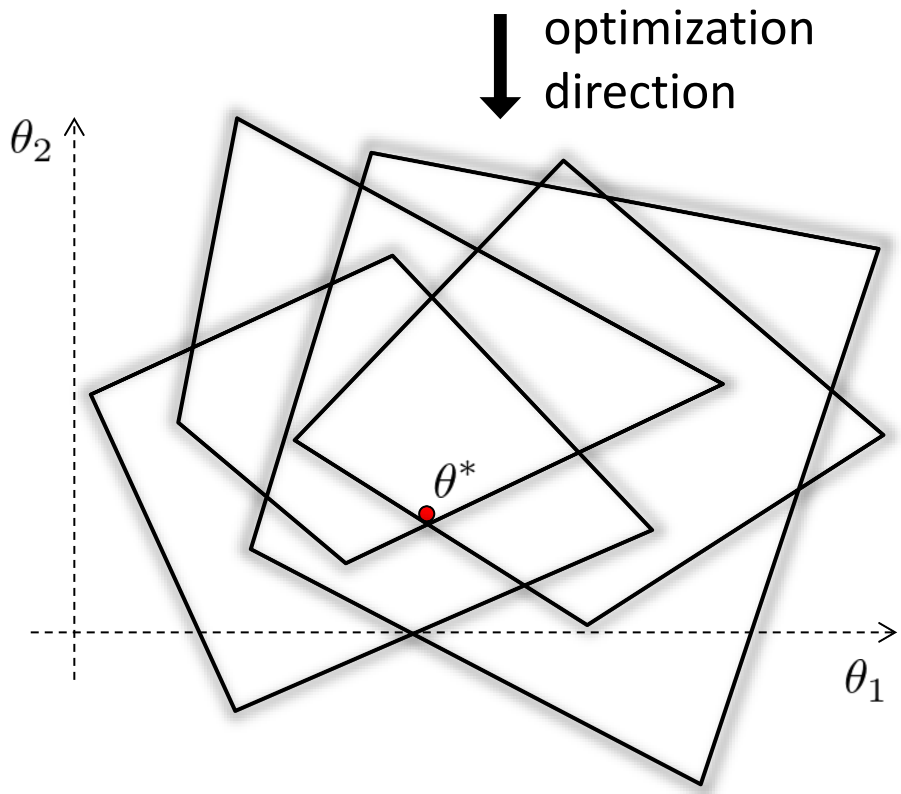
Main goal

➡ **assess $\mathrm{R}(\mathcal{H})$ from data, the same used for design**

Why not using new data for validation:

- using some data for testing rather than designing is a waste of information!
- scenarios (data) are often limited resources (collecting data can be time-consuming or burdensome, involving a monetary cost)
- in many contexts validation is not necessary… **data can play well a double role!**

Risk assessment via sample compression



Compression function

$$\kappa(\delta_1, \ldots, \delta_N) = \delta_{i_1}, \ldots, \delta_{i_k}$$

map extracting a subsample
from a sample of scenarios

Preference

$$\kappa(\delta_1, \ldots, \delta_N) \subseteq S \subseteq (\delta_1, \ldots, \delta_N)$$

$$\Downarrow$$

$$\kappa(S) = \kappa(\delta_1, \ldots, \delta_N)$$

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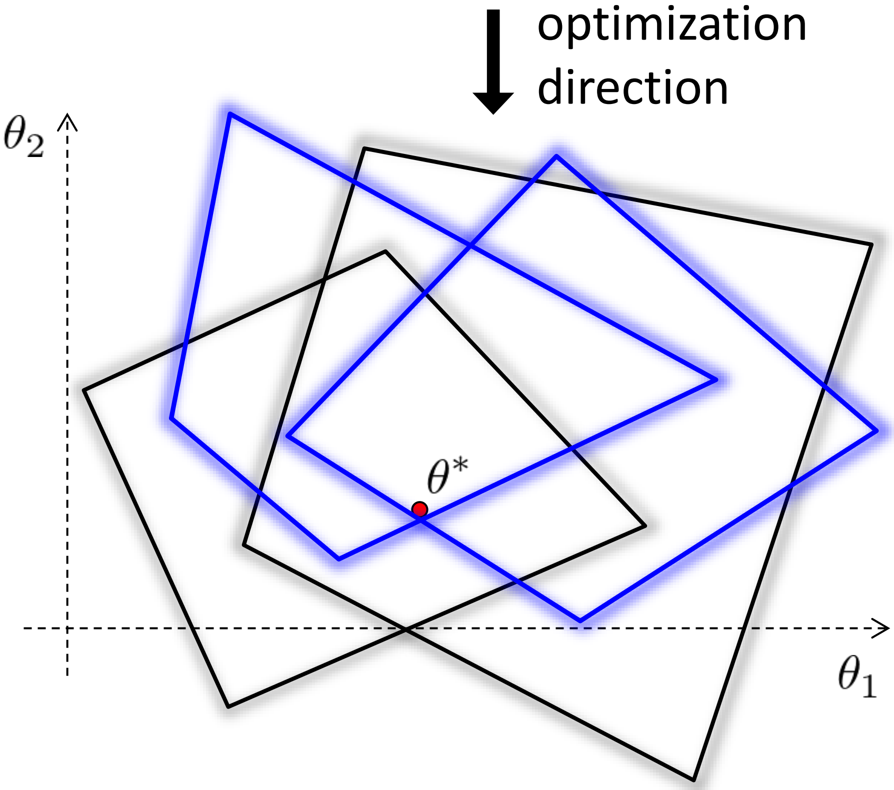

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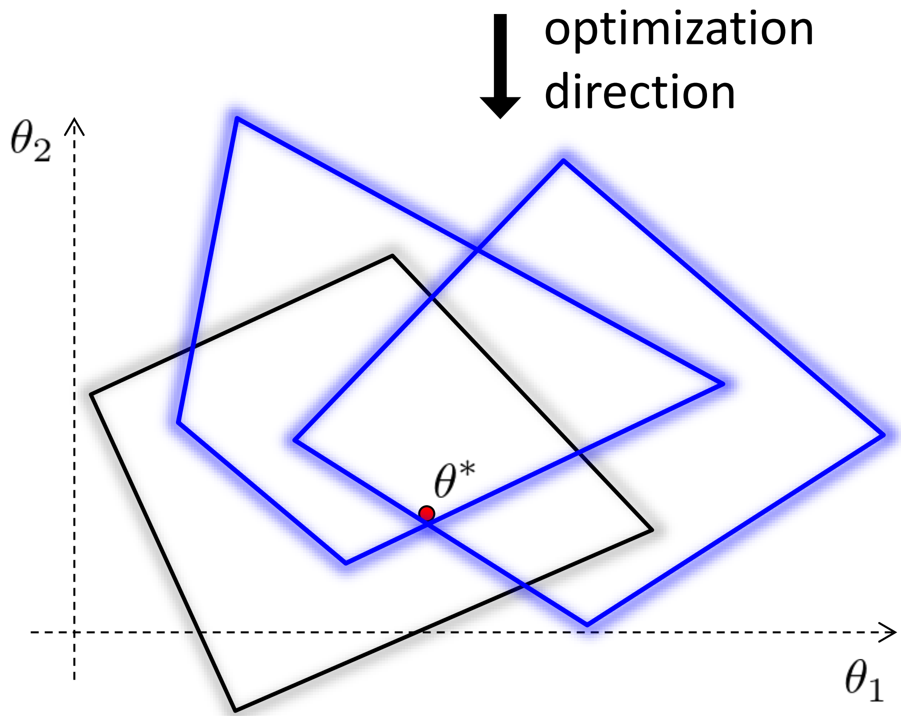

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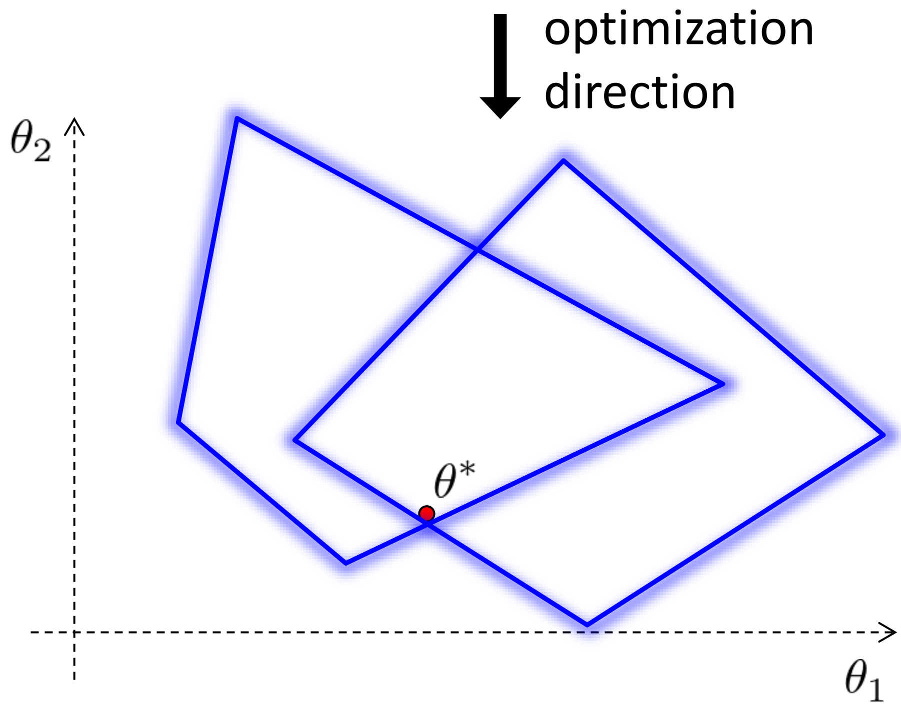


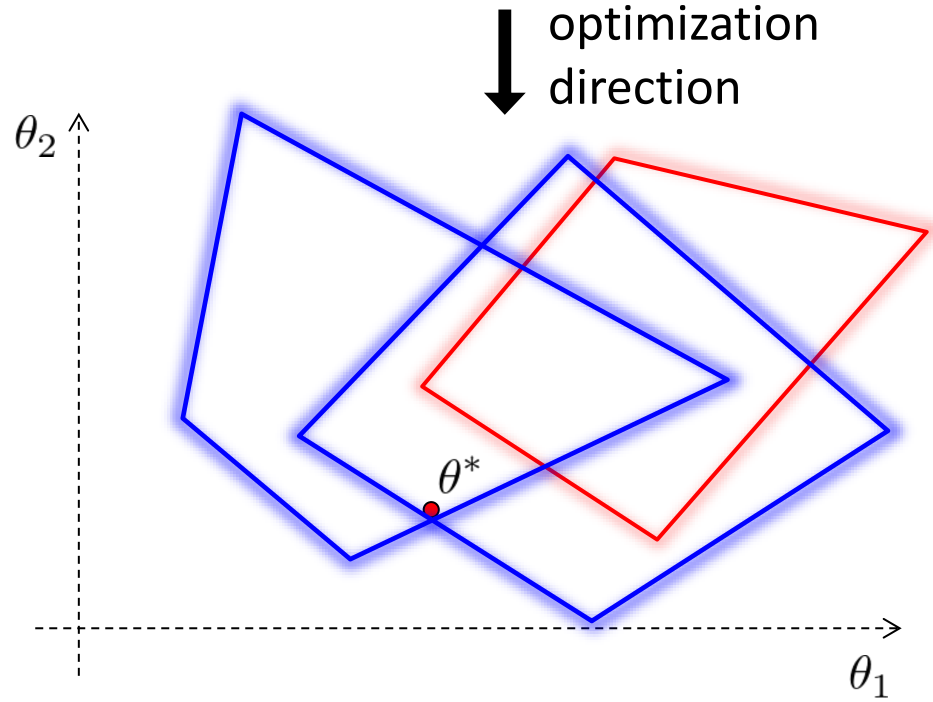
Coherence

a new scenario for which $\mathcal{H}$ is inappropriate is added

$\Downarrow$

the compression must change

Risk assessment via sample compression



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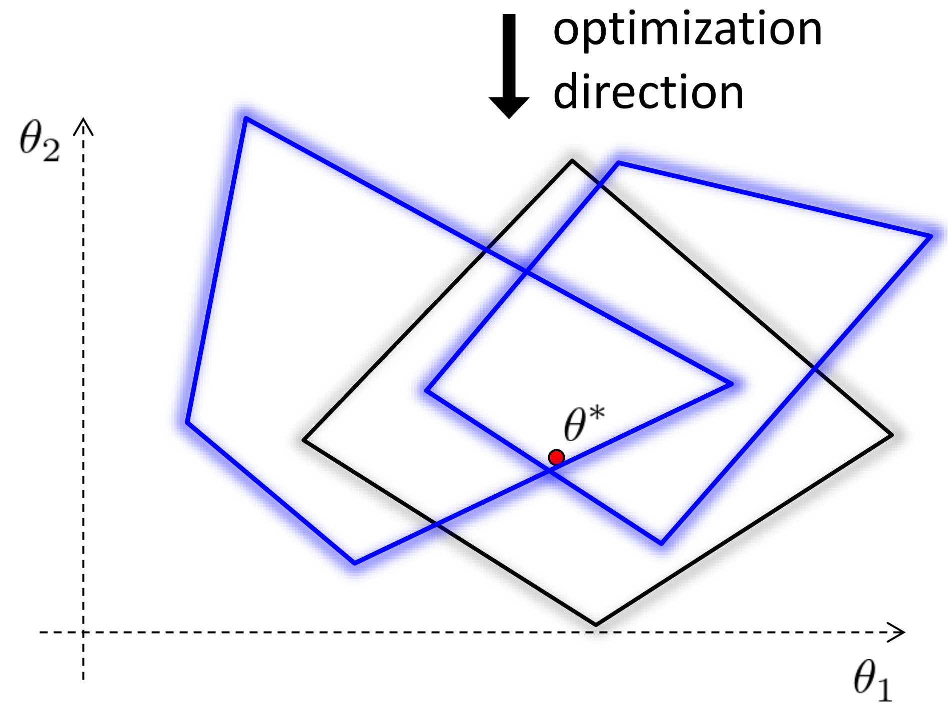

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The main result in a nutshell

Risk: $\mathsf{R}(\mathcal{H}) = \mathsf{R}\big(\mathcal{H}(\delta_1, \ldots, \delta_N)\big)$

Complexity: $\pi = \big|\kappa(\delta_1, \ldots, \delta_N)\big|$

} random variables

↑ size of compressed set

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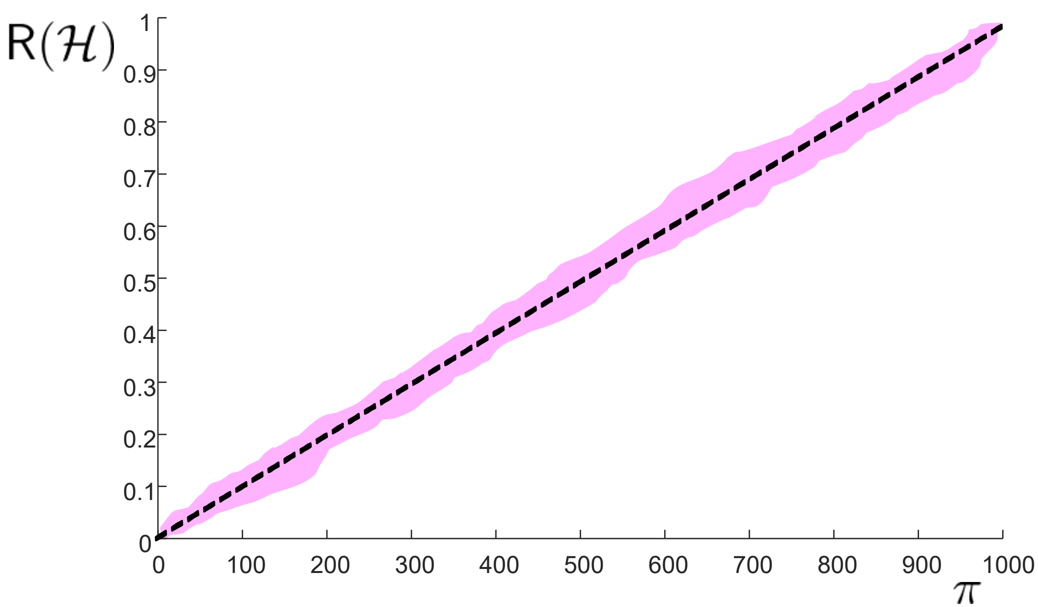
Under preference and coherence, the joint distribution of risk and complexity is concentrated around/below $\mathsf{R}(\mathcal{H}) = \pi/N$



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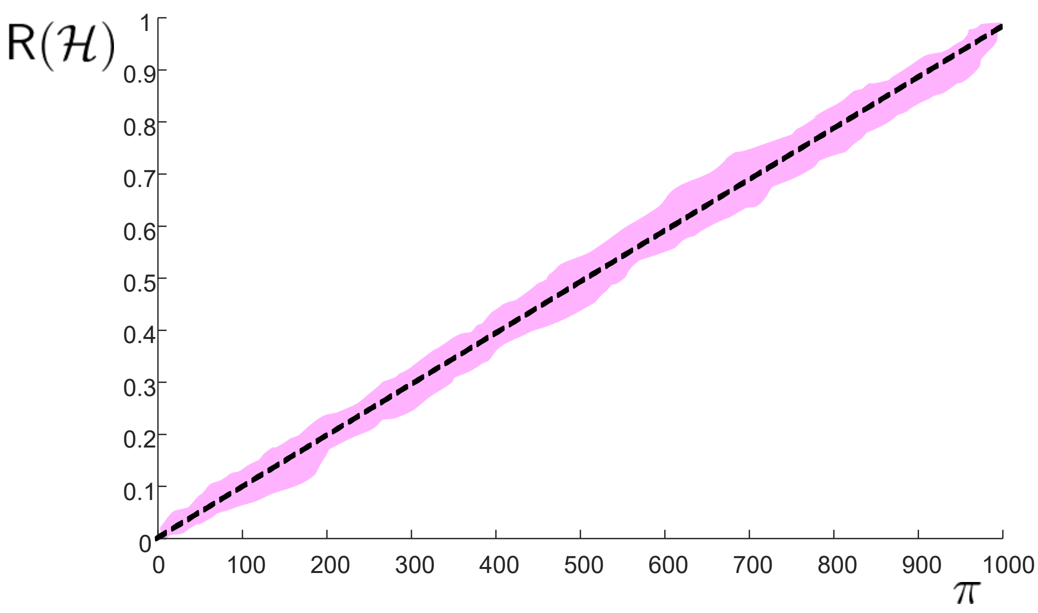

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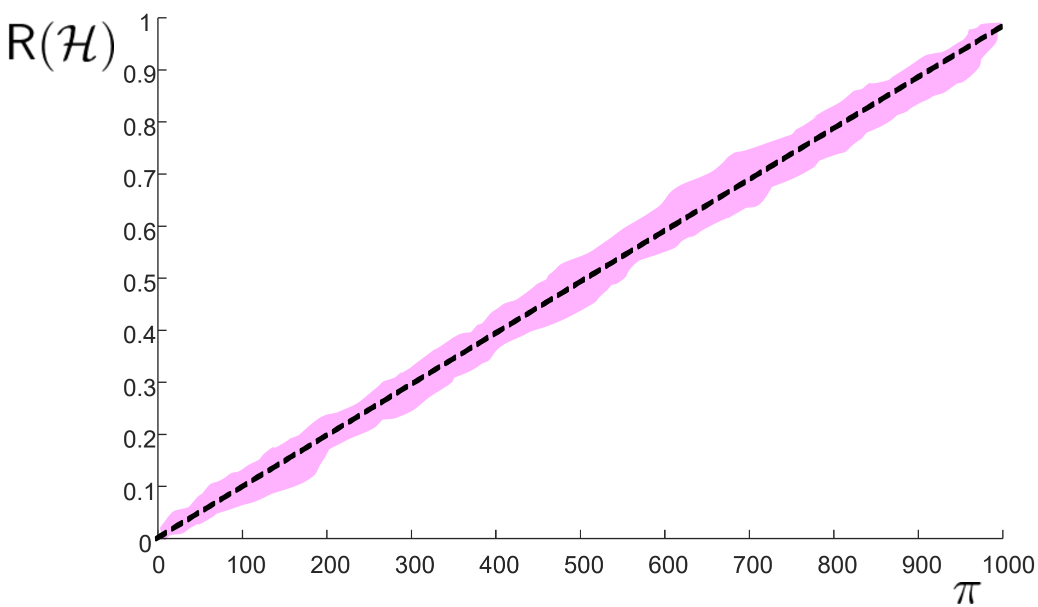

$\mathsf{R}(\mathcal{H})$ can be accurately estimated from π

observable!

Main result (cont'd)

Theorem (with M. Campi)

Assume preference and coherence

Choose $\beta \in (0,1)$ (confidence parameter)

Let $\epsilon_L(k), \epsilon^U(k)$ be the unique roots in (0,1) of polynomials

- $$\binom{N}{k}(1-\epsilon)^{N-k} - \frac{\beta}{2N}\sum_{m=k}^{N-1}\binom{m}{k}(1-\epsilon)^{m-k}$$
- $$\binom{N}{k}(1-\epsilon)^{N-k} - \frac{\beta}{2N}\sum_{m=N+1}^{2N}\binom{m}{k}(1-\epsilon)^{m-k}$$

Then, irrespective of $\mathbb{P}$ (distribution-free),

$$\mathbb{P}^N\left\{\delta_1,\ldots,\delta_N : \epsilon^L(\pi) \leq \mathsf{R}(\mathcal{H}) \leq \epsilon^U(\pi)\right\} \geq 1-\beta$$

Main result (cont'd)

true with confidence $1 - \beta$

claim: $\epsilon_L(\pi) \leq \mathsf{R}(\mathcal{H}) \leq \epsilon^U(\pi)$

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close each other even with finite N,
gap goes to zero as $1/\sqrt{N}$

Main result (cont'd)

true with confidence $1-\beta$

claim: $\epsilon_L(\pi) \leq \mathbf{R}(\mathcal{H}) \leq \epsilon^U(\pi)$

Complexity is a universal observable to obtain very informative assessments of the actual risk !

with finite N,

$/\sqrt{N}$

$\mathrm{R}(\mathcal{H})$ ⇒ not accessible

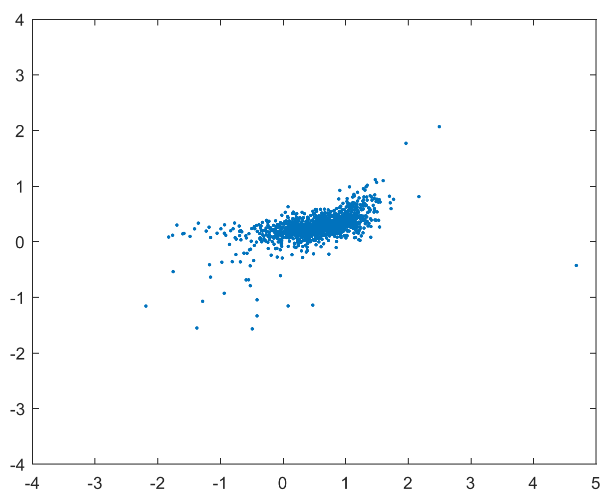
$[\epsilon_L(\pi), \epsilon^U(\pi)]$ ⇒ accessible

- accept/reject the solution
- make further decisions
- compare various "decisions"

Example: reachability analysis via SVDD

$$\begin{cases} x(t+1) = f_\delta\big(x(t), w_\delta(t)\big) \\ x(0) = \bar{x}_\delta \end{cases}$$

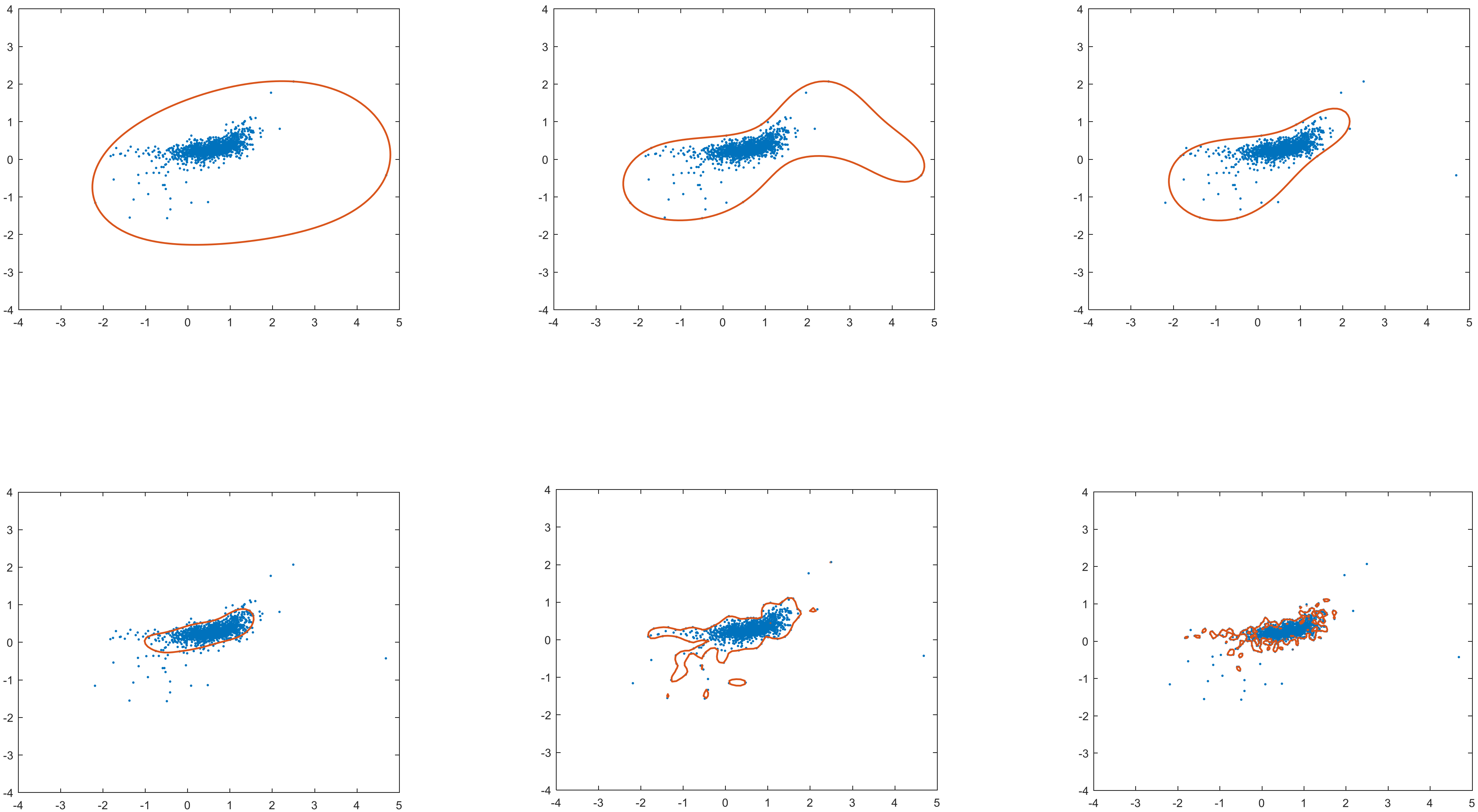
Goal *(Arcak, Devonport, Dietrich, Tu)* : construct a reachable set S such that the terminal state $x(\mathrm{T})$ lies in S with a prescribed probability



SVDD → S

hyperparameters

Example: reachability analysis via SVDD



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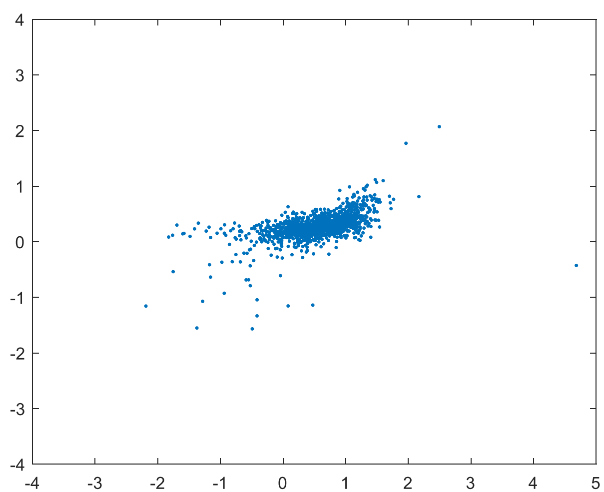

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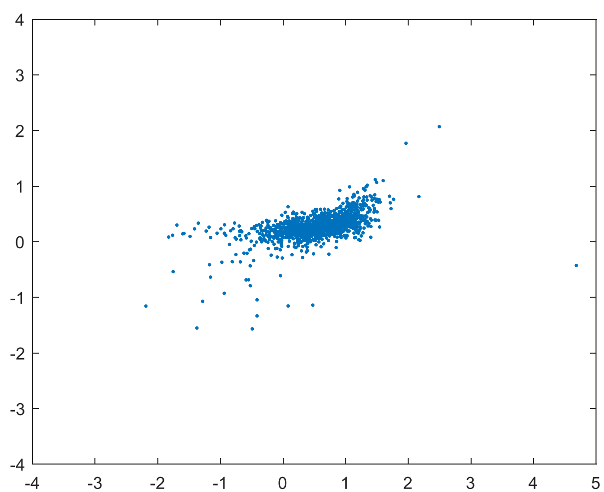
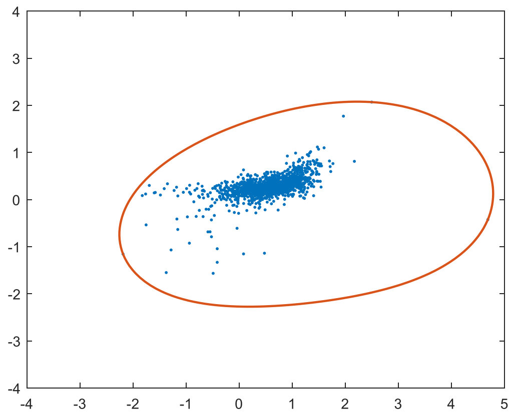


Compression = Support Vectors

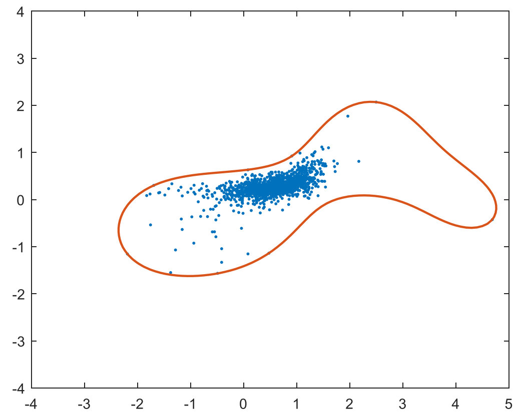
SVDD → S

hyperparameters

Example: reachability analysis via SVDD



$0\% \leq \mathsf{R}(\mathcal{H}) \leq 1.9\%$



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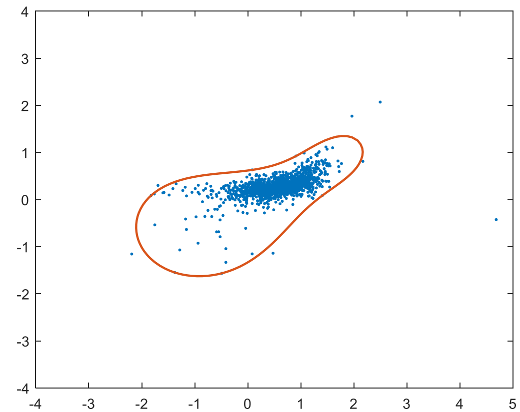
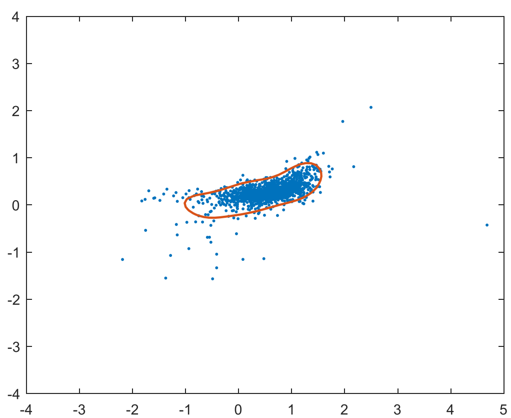
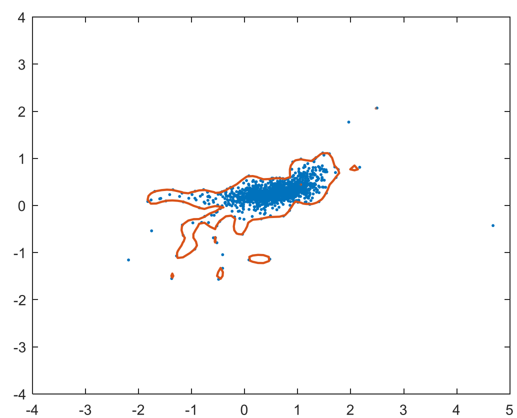


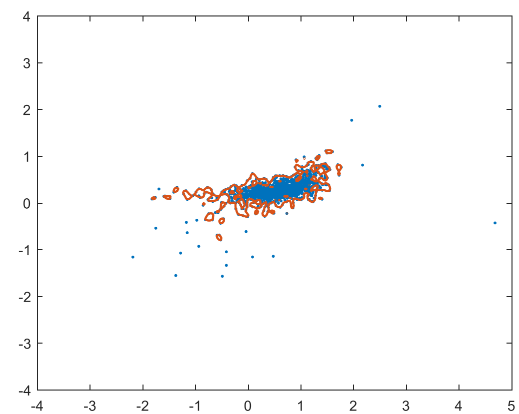
$0\% \leq \mathsf{R}(\mathcal{H}) \leq 2.8\%$



$4.3\% \leq \mathsf{R}(\mathcal{H}) \leq 11.2\%$

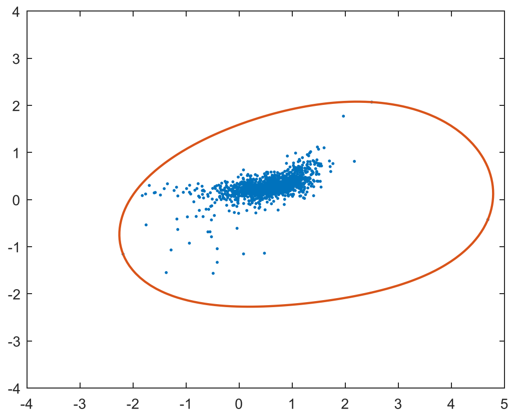


$2.6\% \leq \mathsf{R}(\mathcal{H}) \leq 8.5\%$



$12.4\% \leq \mathsf{R}(\mathcal{H}) \leq 22.7\%$

Example: reachability analysis via SVDD



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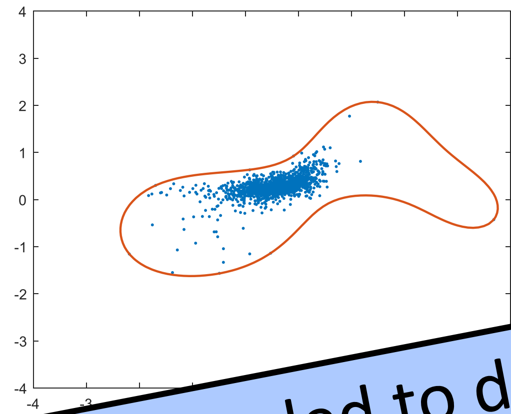
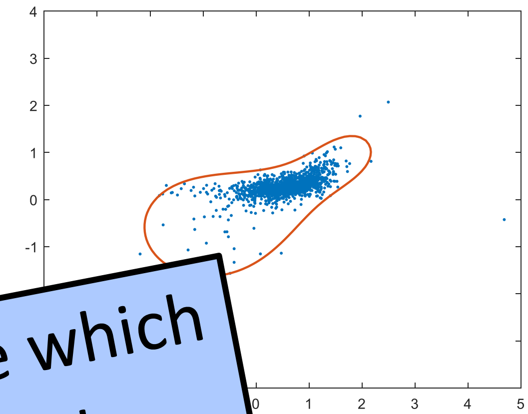
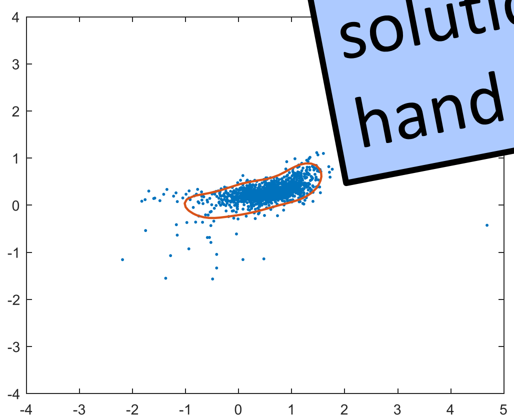




$(\mathcal{H}) \leq 2.8\%$

The information needed to decide which solution is best for the application at hand is provided !!



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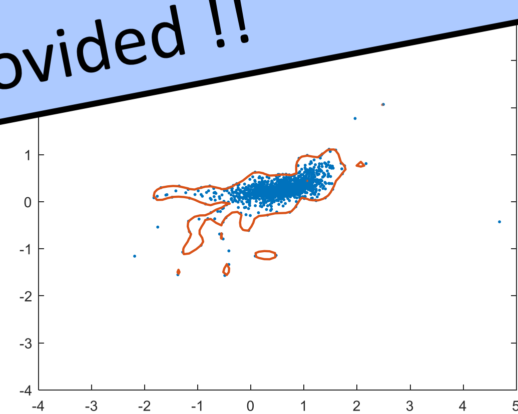

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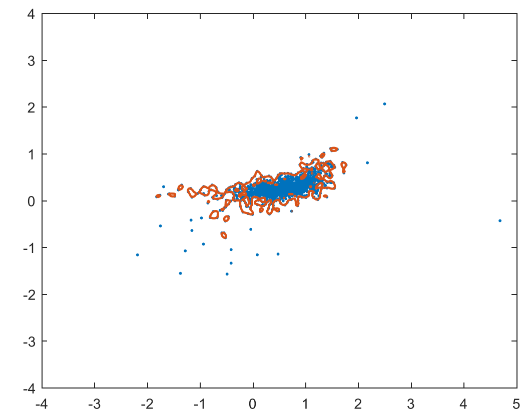

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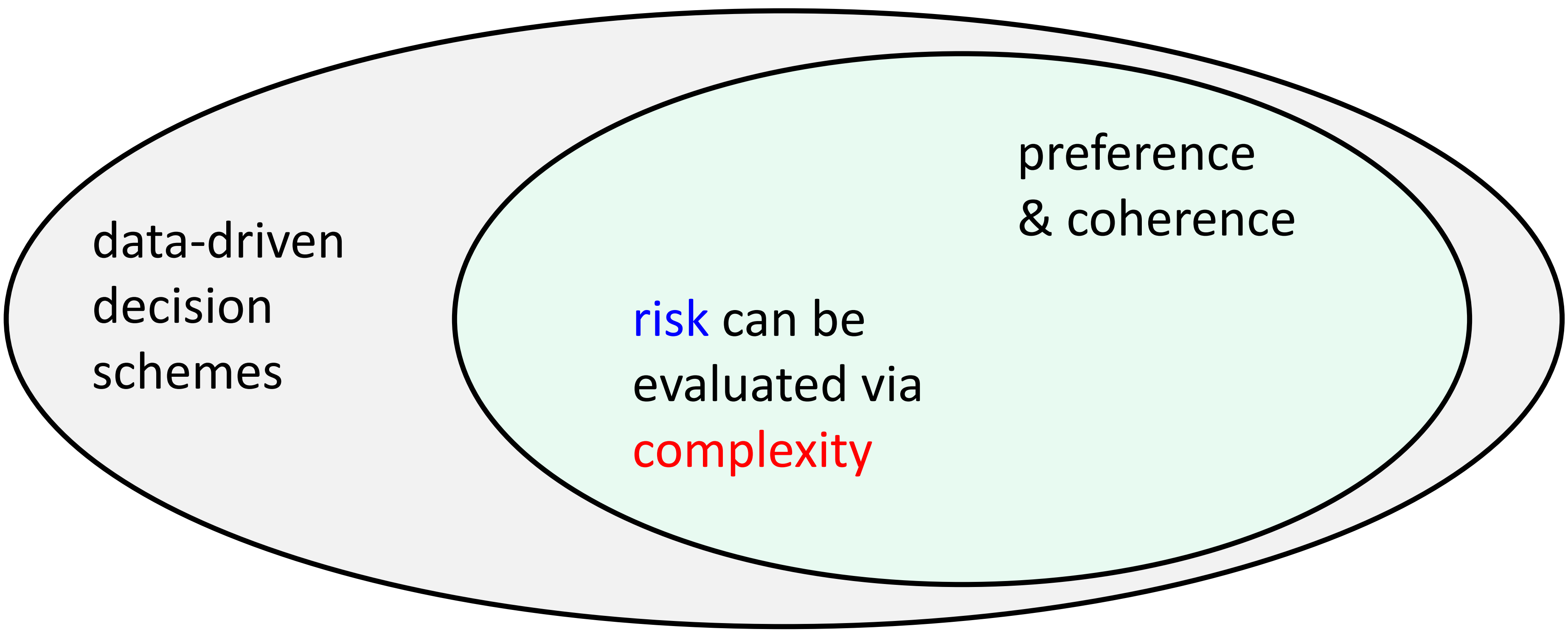
Scenario Approach: range of applicability

Many decision schemes (all scenario optimization schemes, many schemes in ML…) naturally satisfy compression properties… many yet to be discovered…

However, many others do not… notably: SGD



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see also ***WeC02.4***

↑

Idea – the Pick-to-Learn (P2L) algorithm:

a meta-algorithm that builds on an existing data-driven decision scheme as a block-box to induce the compression properties

The Pick-to-Learn (P2L) algorithm

INPUT: scenarios $\delta_1, \delta_2, \ldots, \delta_N$, decision algorithm $\mathfrak{L}$,
initial decision $\mathcal{H}_0$

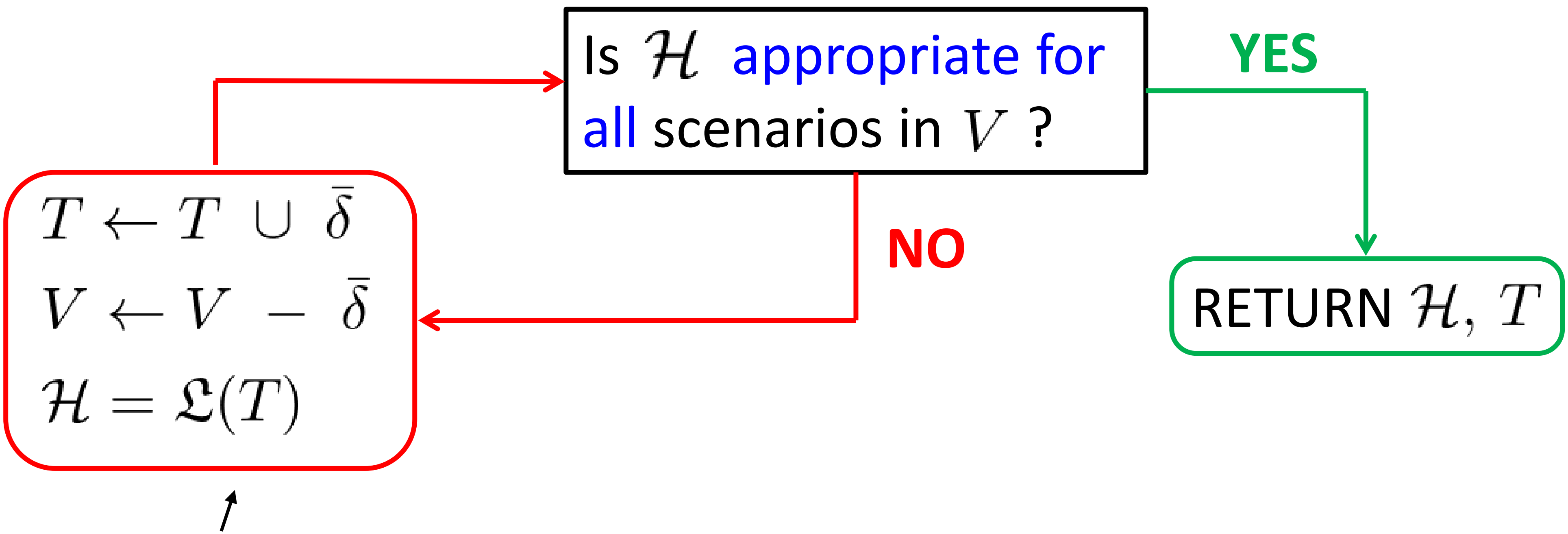
Possibly, **not** linkable to any meaningful compression

theory of the scenario approach **cannot** be directly used to evaluate the risk

The Pick-to-Learn (P2L) algorithm

INPUT: scenarios $\delta_1, \delta_2, \ldots, \delta_N$, decision algorithm $\mathfrak{L}$, initial decision $\mathcal{H}_0$

Initialization: $T = \emptyset,\ V = (\delta_1, \ldots, \delta_N),\ \mathcal{H} = \mathcal{H}_0$



$\bar{\delta}$ = element in V for which $\mathcal{H}$ is most inappropriate

P2L: main features

P2L: $\delta_1, \ldots, \delta_N \to \mathcal{H}$

⇨ new data-driven decision scheme $\mathfrak{L}'$

P2L: $\delta_1, \ldots, \delta_N \to T$

⇨ compression function κ' associated to $\mathfrak{L}'$

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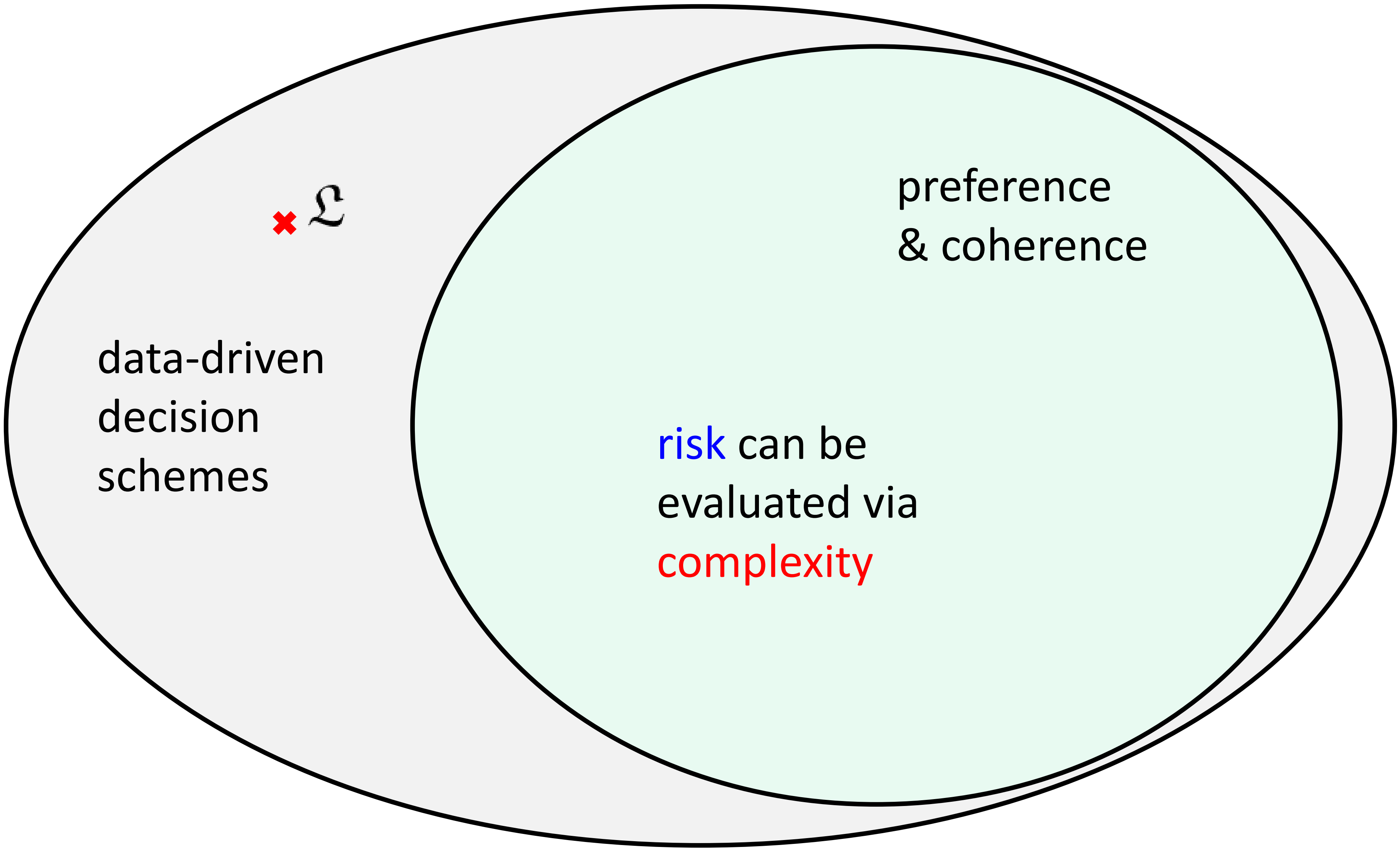
Theorem (with D. Paccagnan and M. Campi)

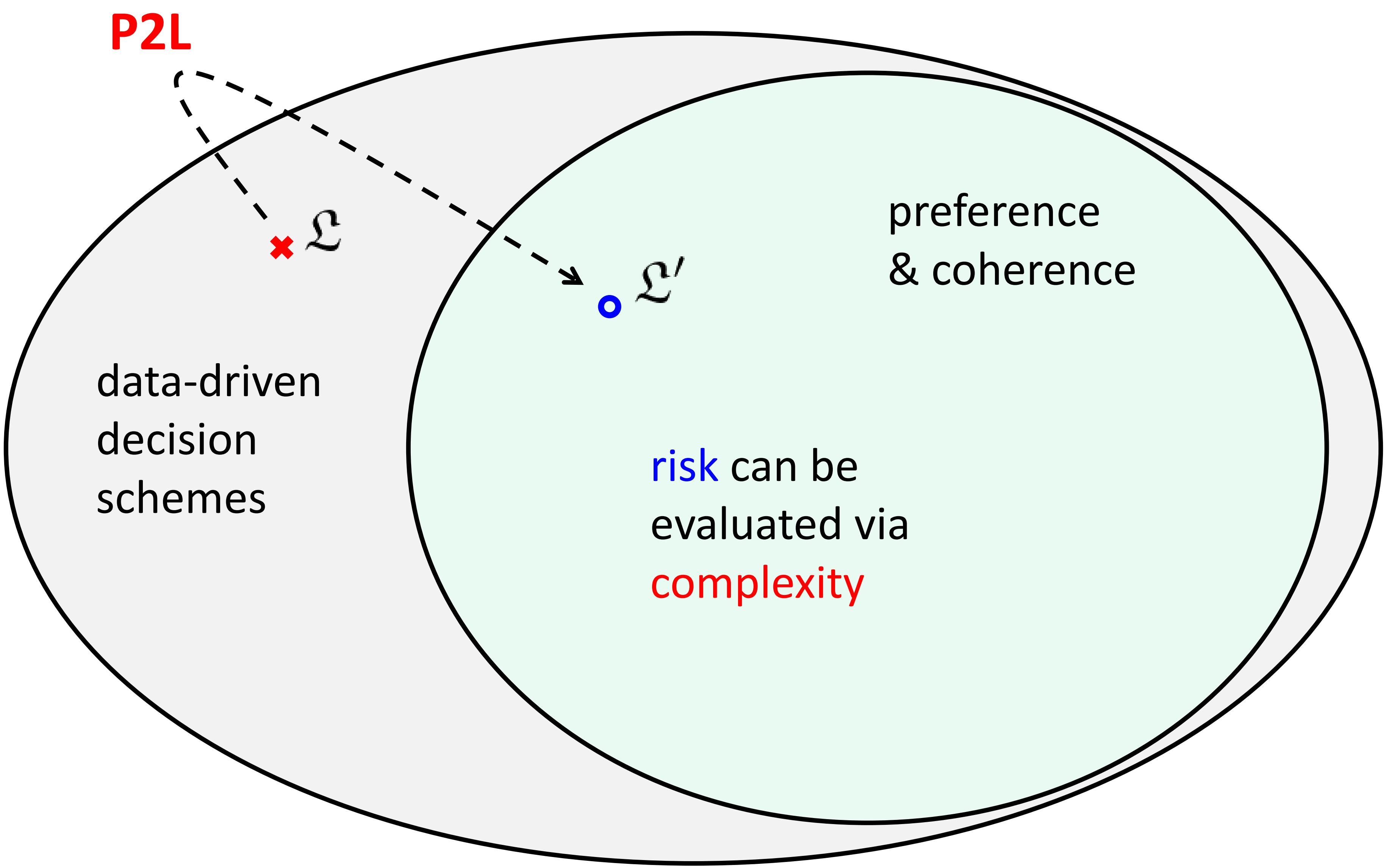
Preference and coherence hold true!

⇨ the risk of $\mathcal{H} = \mathfrak{L}'(\delta_1, \ldots, \delta_N)$ can be assessed via the size of T

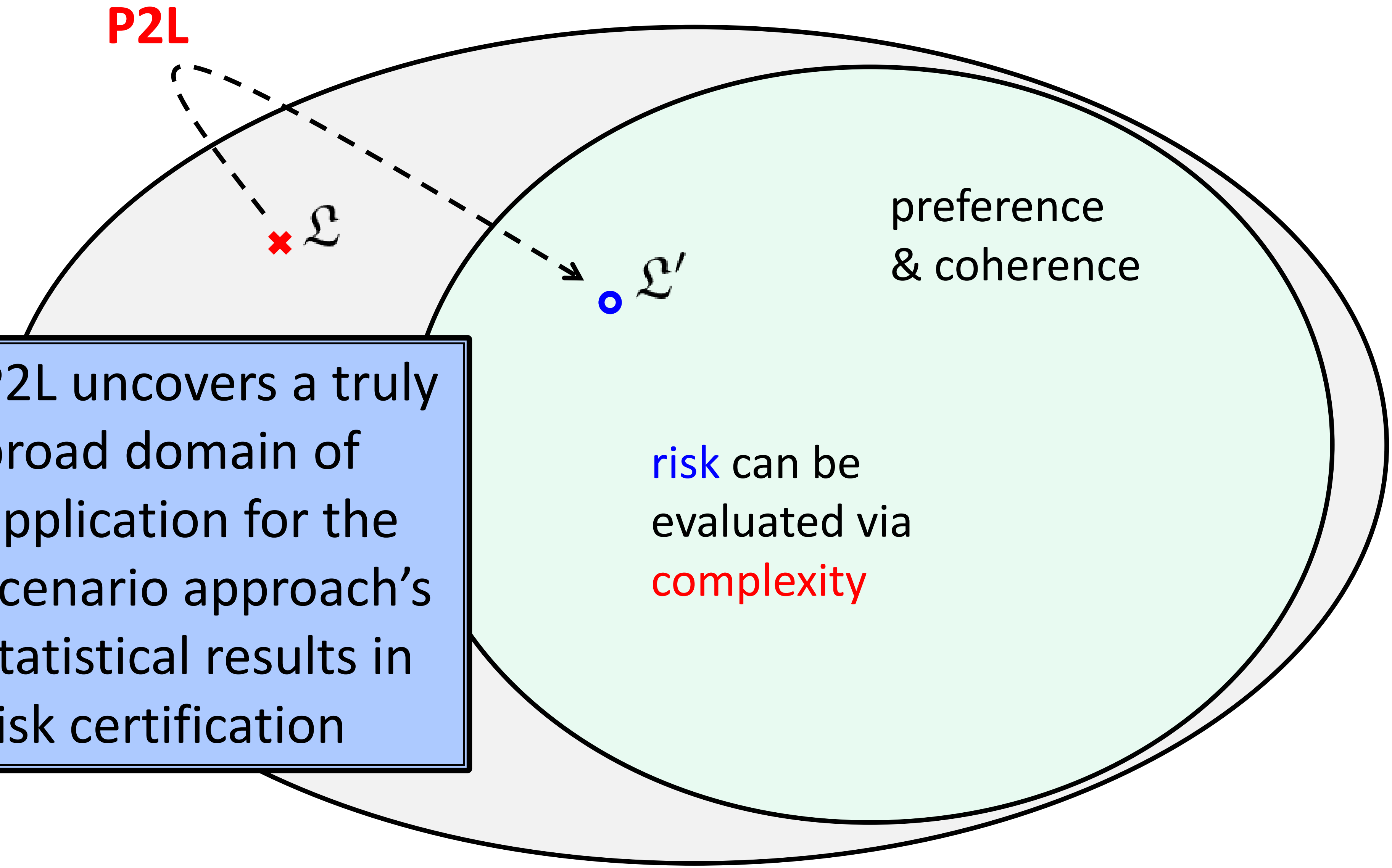
Scenario Approach: range of applicability (cont'd)



Scenario Approach: range of applicability (cont'd)



Scenario Approach: range of applicability (cont'd)



P2L uncovers a truly broad domain of application for the scenario approach's statistical results in risk certification

Thank you !

Relevant articles:

- *M.C. Campi, S. Garatti.* Compression, Generalization and Learning. *Journal of Machine Learning Research,* 24(339):1-74, 2023.

- *D. Paccagnan, M.C. Campi, S. Garatti,* The Pick-to-Learn Algorithm: Empowering Compression for Tight Generalization Bounds and Improved Post-training Performance. *In: Advances in Neural Information Processing Systems 36 (NeurIPS 2023),* 2023.

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