Event-triggered control from data

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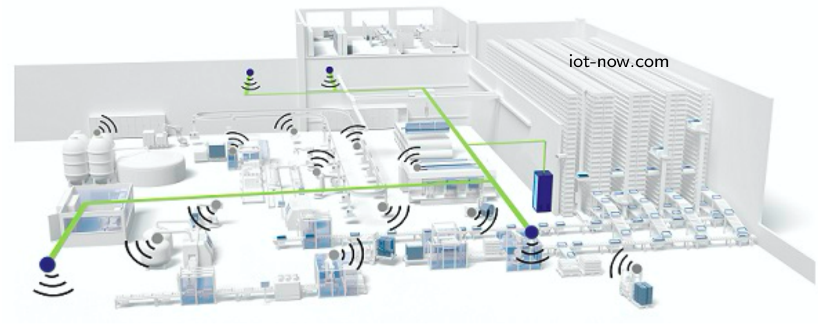
Joint work with C. De Persis (Univ. Groningen) and R. Postoyan (Univ. Lorraine)

Introduction

Event-triggered control from data

ETC is a paradigm for resource-aware control in embedded/networked systems aiming to perform control tasks only when necessary

ETC is predominantly model-based



Learning to control and to communicate from data

We explore how to design ETC directly from data, a step towards **automating** the design of networked control systems

Outline

- Problem formulation
- Learning from noise-free data
- Learning from noisy data
- Extensions and open problems

Problem formulation

Continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t) \quad t \geq 0$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, d is a bounded disturbance.

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where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, d is a bounded disturbance.

GOAL: Design an ETC law

$$u(t) = Kx(t_k), \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N},$$

namely a controller gain K and an event generator for $\{t_k\}$ (*when* u should be updated).

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Well understood when A, B **are known** (Tabuada, Heemels, Nešić, Lemmon, Johansson,...)

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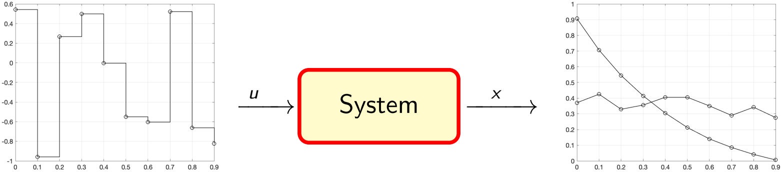
Problem (Event-triggered control from data)

Suppose A, B, d unknown. *Design an ETC law from a dataset*

$$\mathbb{D} = \{\, u(t), x(t), \dot{x}(t), \quad t = 0, T_s, \ldots, (T-1)T_s \,\}$$

collected from the system with an experiment; T = No. samples, T_s = sampling time

Experiment



$$\begin{bmatrix} \text{input} \\ \hline \text{state} \\ \hline \text{state der.} \end{bmatrix} = \begin{bmatrix} U_0 \\ \hline X_0 \\ \hline X_1 \end{bmatrix} := \underbrace{\begin{bmatrix} 0.54 & -0.95 & 0.26 & 0.49 & -0.00 & -0.55 & -0.60 & 0.52 & -0.66 & -0.82 \\ \hline 0.37 & 0.42 & 0.32 & 0.35 & 0.40 & 0.40 & 0.35 & 0.28 & 0.34 & 0.27 \\ 0.90 & 0.70 & 0.54 & 0.41 & 0.30 & 0.21 & 0.14 & 0.08 & 0.04 & 0.00 \\ \hline 0.54 & -0.95 & 0.26 & 0.49 & -0.00 & -0.55 & -0.60 & 0.52 & -0.66 & -0.82 \\ -2.18 & -1.83 & -1.41 & -1.18 & -1.01 & -0.83 & -0.63 & -0.46 & -0.42 & -0.28 \end{bmatrix}}_{\text{dataset } \mathbb{D}}$$

Use the dataset to design K and $\{t_k\}$ (an event generator)

Learning from noise-free data

Learning from noise-free data

An emulation approach (G. Walsh)

We mimic what is typically done in a model-based setting.

Define the error due to sampling

$$e(t) = x(t_k) - x(t), \quad t \in [t_k, t_{k+1})$$

and rewrite the dynamics

$$\dot{x}(t) = Ax(t) + BKx(t_k) = \underbrace{(A + BK)x(t)}_{\text{ideal dynamics}} + \underbrace{BKe(t)}_{\text{error}}$$

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Sequential design:

- first design K ignoring the network
- then design $\{t_k\}$ (event generator) to keep e 'small'

Overall, the closed-loop dynamics will resemble the ideal dynamics $\Longrightarrow$ **emulation**

Lemma (Jan C. Willems et al., 2005)

—For LTI systems, if u is PE the I/O data are equally good as the parametric model—

Consider a discrete-time controllable system $x^+ = Ax + Bu$, *where* $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$. *If* u *is persistently exciting of order* $n+1$ [a] *then the matrix*

$$W = \left[\begin{array}{cccc} u(0) & u(1) & \cdots & u(T-1) \\ \hline x(0) & x(1) & \cdots & x(T-1) \end{array}\right]$$

has full row rank $(n+m)$.

[a] A sequence $u(0), \ldots, u(T-1)$ is persistently exciting (PE) of order L if the matrix

$$\mathcal{H}_u(L) = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-L) \\ u(1) & u(2) & \cdots & u(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \end{bmatrix}$$

has full row rank.

Controller design

Key data-based relation

Data collection Consider system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Run an experiment from initial condition $x(0)$ forced by signal u.

Controller design

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Run an experiment from initial condition $x(0)$ forced by signal u.

Store data into the matrices X_1, X_0, U_0, which satisfy the identity $X_1 = AX_0 + BU_0$:

$$\underbrace{\begin{bmatrix} \dot{x}(0) & \dot{x}(T_s) & \dots & \dot{x}((T-1)T_s) \end{bmatrix}}_{X_1}$$

$$= A \underbrace{\begin{bmatrix} x(0) & x(T_s) & \dots & x((T-1)T_s) \end{bmatrix}}_{X_0} + B \underbrace{\begin{bmatrix} u(0) & u(T_s) & \dots & u((T-1)T_s) \end{bmatrix}}_{U_0}$$

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$$X_1 = AX_0 + BU_0 = \begin{bmatrix}B & A\end{bmatrix}\begin{bmatrix}U_0 \\ X_0\end{bmatrix}$$

Under PE condition of order $n+1$, rank $\begin{bmatrix}U_0 \\ X_0\end{bmatrix} = n + m$

Controller design

Open-loop parametrization of the dynamics

- The least-squares problem

$$\min_{A,B} \left\| X_1 - \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \right\|_F$$

has unique solution

$$\begin{bmatrix} B & A \end{bmatrix} = X_1 \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger$$

- Performing an SVD of

$$\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = U \Sigma V^\top$$

one obtains

$$\begin{bmatrix} B & A \end{bmatrix} = X_1 \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger = X_1 V \Sigma^{-1} U^\top$$

which is the popular "Dynamic mode decomposition" used to estimate A, B from data

- The open-loop data-based model of the system can be used for any control design
- The result belongs to the class of subspace identification modes

J.L. Proctor, S.L. Brunton, J.N. Kutz. "Dynamic mode decomposition with control." *SIAM Journal on Applied Dynamical Systems*, 2016.
M. Verhaegen, V. Verdult. *Filtering and system identification: a least squares approach*. Cambridge University Press, 2007.

Controller design

Closed-loop parametrization of the dynamics

For any K (hypothetically, K must be designed), solve wrt $T \times n$ matrix G:

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

This implies

$$A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} = \overbrace{\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}}^{X_1 = AX_0 + BU_0} G = X_1 G$$

which gives a **closed-loop parametrization of the dynamics**.

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Data-dependent Lyapunov stability

$$(A + BK)P + P(A + BK)^\top + Q < 0, \quad P, Q > 0$$
$$\Updownarrow$$
$$\boxed{(X_1 G)P + P(X_1 G)^\top + Q < 0}$$

C. De Persis, P. Tesi. "Formulas for Data-driven Control: Stabilization, Optimality, and Robustness." *IEEE Transactions on Automatic Control*, 2020.

Controller design

Lemma (Data-based controller design)

Consider a stabilizable linear system $\dot{x} = Ax + Bu$ *and assume that* $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$ *is full row rank. Fix* $Q > 0$ *and consider the data-dependent program:*

$$\textit{find}_{P>0,G,K} \quad X_1 GP + (X_1 GP)^\top + Q < 0$$

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$$

The program is feasible and any solution ensures $A + BK$ *stable.*

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Convexifiable with change of variable $GP = Y$:

$$\begin{aligned} \text{find}_{P>0,Y} \quad & X_1 Y + (X_1 Y)^\top + Q < 0 \\ & P = X_0 Y \\ \text{set} \quad & K = U_0 Y P^{-1} \end{aligned}$$

Learning a triggering policy

Model-based approach

We now account for the network:

$$\dot{x}(t) = \underbrace{(A + BK)x(t)}_{\text{ideal dynamics}} + \underbrace{BK e(t)}_{\text{error}}$$

with $e(t) = x(t_k) - x(t)$, $t \in [t_k, t_{k+1})$.

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Event-triggered policy (Tabuada, 2007)

Take $\underline{V = x^\top P^{-1} x}$ as Lyapunov function for $\dot{x} = (A + BK)x$:

$$\frac{\partial V}{\partial x}((A + BK)x + BKe) \leq \underbrace{-\alpha|x|^2 + \gamma|e|^2}_{\text{closed loop is ISS wrt } e}$$

Stability follows if $|e| \leq \sigma|x|$ with $\sigma^2 < \alpha/\gamma$.

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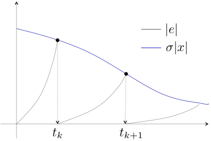

ET policy: $\quad t_{k+1} = \inf\{t > t_k : |e(t)| = \sigma|x(t)|\}$

Learning a triggering policy

Data-based approach

Let K, G be as before, and L solution to

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G, \qquad \begin{bmatrix} K \\ 0_{n \times n} \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} L$$

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$$A + BK = X_1 G \qquad BK = X_1 L$$

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$$A + BK = X_1 G \qquad BK = X_1 L$$

This implies

$$\frac{\partial V}{\partial x}((A+BK)x + BKe) = \begin{bmatrix} x \\ e \end{bmatrix}^\top \underbrace{\begin{bmatrix} (X_1 G)^\top P^{-1} + P^{-1}(X_1 G) & P^{-1}(X_1 L) \\ \star & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ e \end{bmatrix}$$

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Further,

$$t_{k+1} = \inf \left\{ t > t_k : \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} -\sigma^2 I & 0 \\ 0 & I \end{bmatrix}}_{\Psi(\sigma)} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = 0 \right\}$$

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Since $\left[\begin{smallmatrix} x(t) \\ e(t) \end{smallmatrix}\right]^\top \Psi(\sigma) \left[\begin{smallmatrix} x(t) \\ e(t) \end{smallmatrix}\right] \leq 0$ **for all times, for stability we just need** $M - \Psi(\sigma) < 0$

Main result

Theorem (ETC from data)

Consider a stabilizable linear system $\dot{x} = Ax + Bu$ *and assume that* $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$ *is full row rank. Let* $K = U_0 Y P^{-1}$ *with* $P > 0$, Y *solution to*

$$X_1 Y + (X_1 Y)^\top + Q < 0, \quad P = X_0 Y. \tag{1}$$

Let $\mu > 0, \sigma > 0$ *be solutions to*

$$\mu M - \Psi(\sigma) < 0. \tag{2}$$

The event-triggered controller:

1. *renders the closed-loop system GAS;*
2. *has a global minimum inter-event time (MIET).*

Main result

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Consider a stabilizable linear system $\dot{x} = Ax + Bu$ *and assume that* $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$ *is full row rank. Let* $K = U_0 Y P^{-1}$ *with* $P > 0, Y$ *solution to*

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The event-triggered controller:

1. *renders the closed-loop system GAS;*
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Proof (GAS). Combine (1)-(2).

(MIET).

We estimate the time τ taken by $\dfrac{|e|}{|x|}$ to reach σ starting from 0:

$$
\begin{aligned}
\frac{d}{dt}\frac{|e(t)|}{|x(t)|} &= \frac{e(t)^\top \dot{e}(t)}{|e(t)||x(t)|} - \frac{x(t)^\top \dot{x}(t)|e(t)|}{|x(t)|^3} \\
&\leq \left(1 + \frac{|e(t)|}{|x(t)|}\right)\frac{|\dot{x}(t)|}{|x(t)|} \\
&\leq \left(1 + \frac{|e(t)|}{|x(t)|}\right)\frac{|(A+BK)x(t) + BKe(t)|}{|x(t)|} \\
&\leq c\left(1 + \frac{|e(t)|}{|x(t)|}\right)^2
\end{aligned}
$$

where $c := \max\{\underbrace{\|A+BK\|}_{=\,X_1 G}, \underbrace{\|BK\|}_{=\,X_1 L}\}$.

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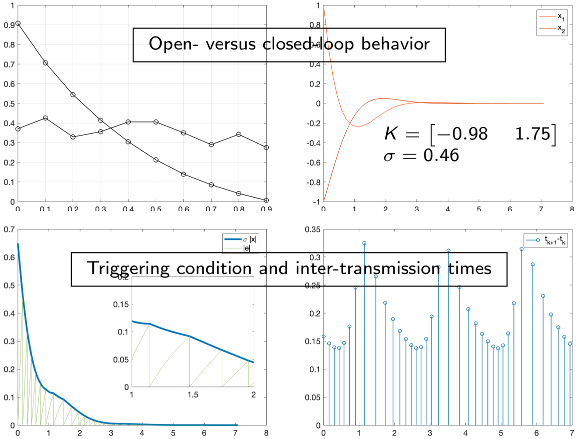
where $c := \max\{\|A + BK\|, \|BK\|\}$, with $A + BK = X_1 G$ and $BK = X_1 L$.

$$\tau \geq \frac{1}{c}\left(\frac{\sigma}{1+\sigma}\right), \text{ and thus } t_{k+1} - t_k \geq \frac{1}{c}\left(\frac{\sigma}{1+\sigma}\right) \text{ for all } k$$

Numerical example

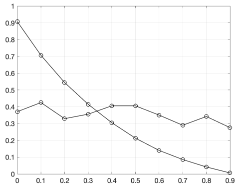
$$A = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} = \left[\begin{array}{cccccccccc} 0.54 & -0.95 & 0.26 & 0.49 & -0.00 & -0.55 & -0.60 & 0.52 & -0.66 & -0.82 \\ \hline 0.37 & 0.42 & 0.32 & 0.35 & 0.40 & 0.40 & 0.35 & 0.28 & 0.34 & 0.27 \\ 0.90 & 0.70 & 0.54 & 0.41 & 0.30 & 0.21 & 0.14 & 0.08 & 0.04 & 0.00 \end{array}\right]$$



ETC from data

Remarks



$\Longrightarrow$

```
cvx_begin sdp
   variable P(n,n) symmetric
   variable Y(T,n)
   X1*Y + (X1*Y)' + Q < 0;
   X0*Y == P;
   P > 0;
cvx_end
K = U0*Y/P;
...
```

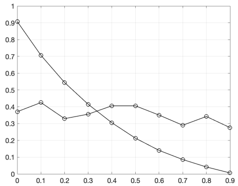
$\Longrightarrow (K, \{t_k\})$

Features of the method:

- End-to-end: from data to controller and triggering policy

ETC from data

Remarks



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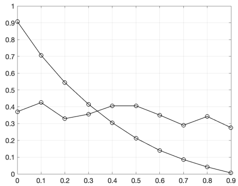
Features of the method:

- End-to-end: from data to controller and triggering policy
- Flexibility: allows for LQR controllers, dynamic triggering rules, etc.

ETC from data

Remarks



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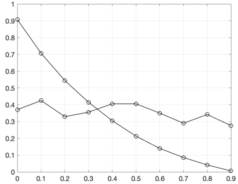
- Flexibility: allows for LQR controllers, dynamic triggering rules, etc.
- Low-complexity in terms of design programs (SDP) and # samples ($T \geq n + m$)

ETC from data

Remarks



$\Longrightarrow$

```
cvx_begin sdp
   variable P(n,n) symmetric
   variable Y(T,n)
   X1*Y + (X1*Y)' + Q < 0;
   X0*Y == P;
   P > 0;
cvx_
K =
...
```

$\Longrightarrow (K, \{t_k\})$

With a large dataset, fix T then **average** (segments of) trajectories of length T. Averaging can be useful with noisy data.

Features of the method:

- End-to-end: from data to controller and triggering policy
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- Low-complexity in terms of design programs (SDP) and # samples ($T \geq n + m$)

Learning from noisy data

Learning from noisy data

Key data-based relation

Data collection Consider now

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t)$$

Run a T-long experiment from initial condition $x(0)$ forced by signals u, d.

Learning from noisy data

Key data-based relation

Data collection Consider now

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t)$$

Run a T-long experiment from initial condition $x(0)$ forced by signals u, d.

Store data into the matrices X_1, X_0, U_0, D_0:

$$\underbrace{\begin{bmatrix}\dot{x}(0) & \dot{x}(T_s) & \dots & \dot{x}((T-1)T_s)\end{bmatrix}}_{X_1}$$
$$= A\underbrace{\begin{bmatrix}x(0) & x(T_s) & \dots & x((T-1)T_s)\end{bmatrix}}_{X_0} + B\underbrace{\begin{bmatrix}u(0) & u(T_s) & \dots & u((T-1)T_s)\end{bmatrix}}_{U_0}$$
$$+ \underbrace{\begin{bmatrix}d(0) & d(T_s) & \dots & d((T-1)T_s)\end{bmatrix}}_{D_0}$$

$$\implies X_1 = AX_0 + BU_0 + D_0 = \begin{bmatrix}B & A\end{bmatrix}\begin{bmatrix}U_0 \\ X_0\end{bmatrix} + D_0$$

We now have

$$A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} = \underbrace{\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}}_{X_1 = AX_0 + BU_0 + D_0} G = (X_1 - D_0)G$$

Lyapunov stability condition

$$\boxed{(X_1 - D_0)G\, P + P\,((X_1 - D_0)G)^\top + Q < 0}$$

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⋆ Same issues with system-ID ⋆

LS has solution $\begin{bmatrix} \hat{B} & \hat{A} \end{bmatrix} = X_1 \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger \Longrightarrow \underbrace{\begin{bmatrix} \hat{B} & \hat{A} \end{bmatrix} - \begin{bmatrix} B & A \end{bmatrix}}_{\text{ESTIMATION ERROR}} = D_0 \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger$

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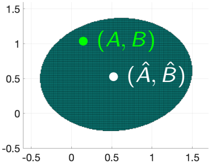
With an INDIRECT approach we could try to build an **uncertainty set** around $(\hat{A}, \hat{B})$ and search for a **robust controller**



Controller design

Accounting for uncertainty

We 'project' directly the uncertainty on the closed-loop dynamics

---

Lyapunov condition

$$(X_1 - D_0)\underbrace{GP}_{Y} + P((X_1 - D_0)G)^\top + Q < 0$$

Controller design
Accounting for uncertainty

We 'project' directly the uncertainty on the closed-loop dynamics

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Lyapunov condition

$$(X_1 - D_0)\underbrace{GP}_{Y} + P((X_1 - D_0)G)^\top + Q < 0$$

Satisfy the Lyap. condition for all D_0 with a certain norm bound:

$$\boxed{(X_1 - D)Y + ((X_1 - D)Y)^\top + Q < 0 \quad \forall D \in \mathcal{D}}$$

where $\mathcal{D} = \left\{D : DD^\top \leq \delta^2 I_n,\ \delta \text{ known}\right\}$.

- if $D_0 \in \mathcal{D}$ we stabilize our system

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- use statistics to get tight values fro δ

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- if $D_0 \in \mathcal{D}$ we stabilize our system
- use statistics to get tight values fro δ
- with statistics, large datasets + averaging can help to reduce δ

Controller design

Petersen's lemma

We want

$$(X_1 - D)Y + ((X_1 - D)Y)^\top + Q < 0 \quad \forall D \in \mathcal{D} \tag{A}$$

Lemma (**Petersen-Hollot**) Let $H = H^\top, M, N$ be given matrices, $\mathcal{D} = \{D : DD^\top \leq \delta^2 I\}$. Then,

$$H - N^\top D M^\top - M D^\top N < 0 \quad \forall D \in \mathcal{D} \tag{B}$$

if and only if there exists $\epsilon > 0$ such that

$$H + \epsilon^{-1} M M^\top + \epsilon \delta^2 N^\top N < 0 \tag{C}$$

Proof (suff.)

$(\sqrt{\epsilon^{-1}} M + \sqrt{\epsilon} N^\top D)(\sqrt{\epsilon^{-1}} M + \sqrt{\epsilon} N^\top D)^\top \geq 0$

We want

$$(X_1 - D)Y + ((X_1 - D)Y)^\top + Q < 0 \quad \forall D \in \mathcal{D} \qquad \text{(A)}$$

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Proof (suff.)

$(\sqrt{\epsilon^{-1}} M + \sqrt{\epsilon} N^\top D)(\sqrt{\epsilon^{-1}} M + \sqrt{\epsilon} N^\top D)^\top \geq 0$

Equation (A) holds if and only if there exists $\epsilon > 0$ such that

$$\underbrace{X_1 Y + (X_1 Y)^\top + Q}_{H} + \epsilon^{-1} Y Y^\top + \epsilon \delta^2 I_n < 0$$

Controller design

Lemma (Robust controller design)

Consider a continuous-time linear system $\dot{x} = Ax + Bu + d$ *and suppose that* $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$ *has full row rank. Let* $D_0 \in \mathcal{D} = \left\{ D : DD^\top \leq \delta^2 I_n,\ \delta \text{ known} \right\}$*, and suppose that there exist* $P > 0, \epsilon > 0$, Y *solution to*

$$\begin{bmatrix} X_1 Y + (X_1 Y)^\top + Q + \epsilon\delta^2 I_n & Y^\top \\ Y & -\epsilon I \end{bmatrix} < 0, \tag{4a}$$

$$P = X_0 Y. \tag{4b}$$

Then, $K = U_0 Y P^{-1}$ *renders* $A + BK$ *stable.*

Controller design

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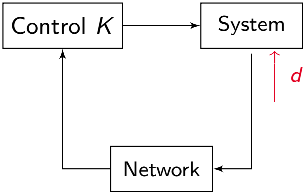
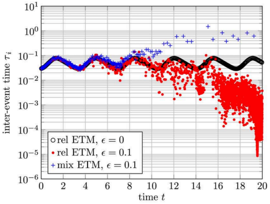
Then, $K = U_0 Y P^{-1}$ *renders* $A + BK$ *stable.*

- a solution may not exist

Learning a triggering policy

Zeno behavior





$|e(t)| \leq \sigma |x(t)|$
no guaranteed MIET
(Borgers, Hemmels, 2015)

Note:

$$\frac{d}{dt}\frac{|e(t)|}{|x(t)|} = \frac{e(t)^\top \dot{e}(t)}{|e(t)||x(t)|} - \frac{x(t)^\top \dot{x}(t)|e(t)|}{|x(t)|^3}$$

$$\leq \left(1 + \frac{|e(t)|}{|x(t)|}\right)\frac{|\dot{x}(t)|}{|x(t)|}$$

$$= \left(1 + \frac{|e(t)|}{|x(t)|}\right)\frac{|(A+BK)x(t) + BKe(t) + d(t)|}{|x(t)|}$$

no upper bound on the growth of $\frac{d}{dt}\frac{|e(t)|}{|x(t)|}$

Mixed (relative-absolute) triggering

$$t_{k+1} = \inf\{t \in \mathbb{R} : t > t_k \text{ and } |e(t)| = \sigma(|x(t)| + \nu)\}$$

$$\begin{aligned}
\frac{d}{dt}\frac{|e(t)|}{|x(t)|+\nu} &= \frac{e(t)^\top \dot{e}(t)}{|e(t)|(|x(t)|+\nu)} - \frac{x(t)^\top \dot{x}(t)|e(t)|}{|x(t)|(|x(t)|+\nu)^2} \\
&\leq \left(1+\frac{|e(t)|}{|x(t)|+\nu}\right)\frac{|\dot{x}(t)|}{|x(t)|+\nu} \\
&= \left(1+\frac{|e(t)|}{|x(t)|+\nu}\right)\frac{|(A+BK)x(t)+BKe(t)+d(t)|}{|x(t)|+\nu} \\
&\leq c\left(1+\frac{|e(t)|}{|x(t)|+\nu}\right)^2
\end{aligned}$$

with $c := \max\{\|A+BK\|, \|BK\|, d_\star/\nu\}$ where $d_\star := \|d\|_\infty$

It guarantees a global MIET $= \dfrac{1}{c}\left(\dfrac{\sigma}{1+\sigma}\right)$

Learning a triggering policy

Data-based approach

Decay of Lyapunov function

$$\frac{\partial V}{\partial x}\big((A+BK)x+BKe+d\big) \leq$$

$$-x^\top \frac{P^{-1}QP^{-1}}{2}x + \begin{bmatrix} x \\ e \end{bmatrix}^\top \underbrace{\begin{bmatrix} -\frac{P^{-1}QP^{-1}}{2} & P^{-1}(X_1 - D_0)L \\ \star & 0 \end{bmatrix}}_{M(D_0)} \begin{bmatrix} x \\ e \end{bmatrix} + 2d^\top P^{-1}x$$

Learning a triggering policy
Data-based approach

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The triggering rule ensures $|e|\leq\sigma(|x|+\nu)$ for all times, thus

$$\begin{bmatrix}x\\e\end{bmatrix}^\top\underbrace{\begin{bmatrix}-2\sigma^2 I_n & 0\\ 0 & I_n\end{bmatrix}}_{\Psi(\sigma)}\begin{bmatrix}x\\e\end{bmatrix}\leq 2\sigma^2\nu^2$$

Learning a triggering policy

Data-based approach

Decay of Lyapunov function

$$\frac{\partial V}{\partial x}((A+BK)x + BKe + d) \leq$$

$$-x^\top \frac{P^{-1}QP^{-1}}{2}x + \begin{bmatrix} x \\ e \end{bmatrix}^\top \underbrace{\begin{bmatrix} -\frac{P^{-1}QP^{-1}}{2} & P^{-1}(X_1 - D_0)L \\ \star & 0 \end{bmatrix}}_{M(D_0)} \begin{bmatrix} x \\ e \end{bmatrix} + 2d^\top P^{-1}x$$

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$$\begin{bmatrix} x \\ e \end{bmatrix}^\top \underbrace{\begin{bmatrix} -2\sigma^2 I_n & 0 \\ 0 & I_n \end{bmatrix}}_{\Psi(\sigma)} \begin{bmatrix} x \\ e \end{bmatrix} \leq 2\sigma^2\nu^2$$

Ensuring $M(D_0) - \Psi(\sigma) < 0$ implies practical ISS

Main result

Theorem (ETC from data)

Consider a continuous-time linear system $\dot{x} = Ax + Bu + d$ *and suppose that* $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$ *has full row rank. Let* $D_0 \in \mathcal{D} = \{D : DD^\top \leq \delta^2 I_n,\ \delta \text{ known}\}$*, and suppose that there exist* $P > 0, \epsilon > 0, Y$ *solution to*

$$\begin{bmatrix} X_1 Y + (X_1 Y)^\top + Q + \epsilon\delta^2 I_n & Y^\top \\ \star & -\epsilon I \end{bmatrix} < 0, , \quad P = X_0 Y$$

Let $\mu, \alpha, \sigma > 0$ *be solution to*

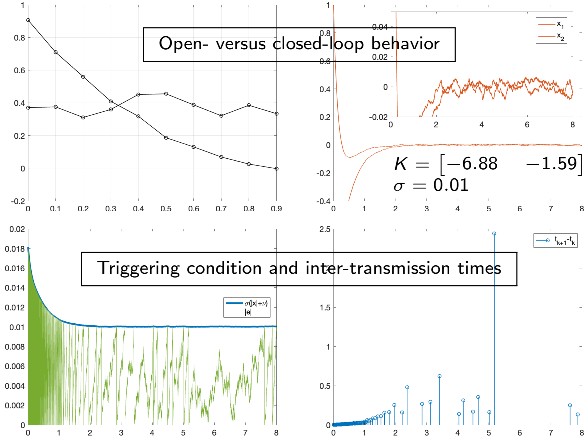
$$\underbrace{\begin{bmatrix} -\dfrac{\mu P^{-1} Q P^{-1}}{2} + 2\sigma^2 I & \mu P^{-1} X_1 L & \mu\delta P^{-1} \\ \star & -I + \alpha L^\top L & 0 \\ \star & \star & -\alpha I \end{bmatrix}}_{\text{ensures } \mu M(D_0) - \Psi(\sigma) < 0} < 0$$

Then the event-triggered controller with $K = U_0 Y P^{-1}$ *ensures practical exponential ISS along with a global MIET.*

As before, few samples may suffice

$$A = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ d \in \mathcal{U}[-0.5, 0.5](6\text{dB}),\ \delta^2 = 2.5,\ \nu = 1$$

$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} = \left[\begin{array}{cccccccccc} 0.54 & -0.95 & 0.26 & 0.49 & -0.00 & -0.55 & -0.60 & 0.52 & -0.66 & -0.82 \\ \hline 0.37 & 0.37 & 0.31 & 0.35 & 0.45 & 0.45 & 0.38 & 0.32 & 0.38 & 0.33 \\ 0.90 & 0.70 & 0.56 & 0.40 & 0.31 & 0.18 & 0.13 & 0.06 & 0.02 & -0.00 \end{array}\right]$$



Extensions and open research questions

Extensions

- general quadratic policies
- dynamic triggering
- ISS via time-regularized triggering

Extensions and open research questions

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- general quadratic policies
- dynamic triggering
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Open problems

- dynamic controllers
- nonlinear systems (ISS)
- large-scale systems

very much interdisciplinary

Reference

C. De Persis, R. Postoyan, P. Tesi. *Event-triggered Control from Data*. IEEE Transactions on Automatic Control (provv. accepted); arXiv:2208.11634

Event-triggered Control From Data

C. De Persis, R. Postoyan, and P. Tesi

***Abstract*—We present a data-based approach to design event-triggered state-feedback controllers for unknown continuous-time linear systems affected by disturbances. By an event, we mean state measurements transmission from the sensors to the**

control, see e.g., [6]–[10] for earlier contributions and one recent survey on the topic. This paradigm is also appealing as it may ease the controller design step. Few techniques are currently available in the literature to design data-driven

Consider the **integral** version of $\dot{x} = Ax + Bu + d$:

$$\int_{t_1}^{t_2} \dot{x}(t)dt = \int_{t_1}^{t_2} (Ax(t) + Bu(t) + d(t))dt,$$

which gives

$$x(t_2) - x(t_1) = A\int_{t_1}^{t_2} x(t)dt + B\int_{t_1}^{t_2} u(t)dt + \int_{t_1}^{t_2} d(t)dt$$

At sampling times:

$$\underbrace{x((k+1)T_s) - x(kT_s)}_{\xi(k)} = A\underbrace{\int_{kT_s}^{(k+1)T_s} x(t)dt}_{r(k)} + B\underbrace{\int_{kT_s}^{(k+1)T_s} u(t)dt}_{v(k)} + \underbrace{\int_{kT_s}^{(k+1)T_s} d(t)dt}_{w(k)}$$

$\Longrightarrow$ $\underline{\Xi = AR + BV + W}$ instead of $X_1 = AX_0 + BU_0 + D_0$

Data processing

Averaging and noise statistics

Suppose we make N experiments and let $\left(U_0^{(r)}, D_0^{(r)}, X_0^{(r)}, X_1^{(r)}\right)$ be the r-th dataset. Then **average**:

$$\underline{X}_1 = A\underline{X}_0 + B\underline{U}_0 + \underline{D}_0 \quad \text{with } \underline{S} := \frac{1}{N}\sum_{r=1}^{N} S^{(r)}$$

If noise has suitable statistics, **averaging reduces noise variance**.

For example, let the noise be i.i.d. drawn from $\mathcal{N}(0,\Sigma)$. For all $\mu > 0$ [1,2]

$$\|\underline{D}_0\| \leq \underbrace{\sqrt{\frac{T}{N}}\left(\lambda_{max}(\Sigma^{1/2})(1+\mu) + \sqrt{\frac{\text{trace}(\Sigma)}{T}}\right)}_{\delta}$$

with probability at least $1 - \exp\left(-T\mu^2/2\right)$.

We can make δ as small as we like by increasing N.

[1] M. Wainwright, *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge University Press, 2019.

[2] Similar bounds for other noise models

S-procedure

Lemma (S-procedure (Yakubovich, 70s))

Let $F_0, F_1 \in \mathbb{S}^{n \times n}$. It holds that

$$z^\top F_0 z \leq 0 \text{ for all } z : z^\top F_1 z \leq 0$$

if there exists $\alpha \geq 0$ such that $F_0 - \alpha F_1 \leq 0$. The converse holds provided there exists a point u with $u^\top F_1 u < 0$.

In our case,

$$\frac{\partial V}{\partial x}((A + BK)x + BKe) = \begin{bmatrix} x \\ e \end{bmatrix}^\top M \begin{bmatrix} x \\ e \end{bmatrix}$$

and the triggering rule ensures

$$\begin{bmatrix} x \\ e \end{bmatrix}^\top \Psi(\sigma) \begin{bmatrix} x \\ e \end{bmatrix} \leq 0 \text{ for all } (x, e) \neq 0$$

The sufficiency part of the S-procedure gives $(M + \epsilon I_n) - \alpha\Phi(\sigma) \leq 0$ for some $\epsilon > 0$, which is equivalent to $\alpha^{-1} M - \Phi(\sigma) < 0$.